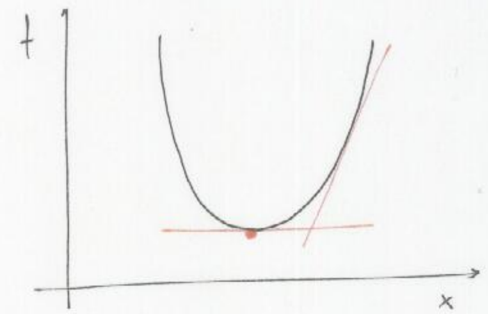
CALCULUS of VARIATIONS:

- LOCAL MINIMA of UNCONSTRAINED FUNCTION

- Theorem: IF $x^* \in \mathbb{R}^n$ is A LOCAL MINIMA for $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ with $f \in \underbrace{C^1}_{}(\mathbb{R}^n)$, then:

CLASS of FUNCTIONS CONTINUOS and DERIVATIVE

$$\frac{\partial f}{\partial x}(x^*) = 0$$



- Theorem (NECESSARY CONDITION): IF x^* is a $\overbrace{\text{MINIMUM}}^{\text{point of}}$ OF THE function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$, and if $f \in C^2(\mathbb{R}^n)$, then

$$\frac{\partial f}{\partial x}(x^*) = 0 \quad \text{and} \quad H_f = \begin{bmatrix} \frac{\partial f^2}{\partial x_1^2} & \cdots & \frac{\partial f^2}{\partial x_1 \partial x_n} \\ \frac{\partial f^2}{\partial x_n \partial x_1} & \cdots & \frac{\partial f^2}{\partial x_n^2} \end{bmatrix} \geq 0$$

- Theorem (SUFFICIENT CONDITION): if $x^* \in \mathbb{R}^n$ is such that:

$$\frac{\partial f}{\partial x}(x^*) = 0 \quad \text{and} \quad Hf(x^*) > 0$$

then x^* is a point of MINIMUM for $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ ■

- LOCAL MINIMA of CONSTRAINED FUNCTIONS:

- $$\min_{x \in \mathbb{R}^n} f(x) \qquad \text{s.t.} \quad g(x) \le 0, \quad h(x) = 0$$

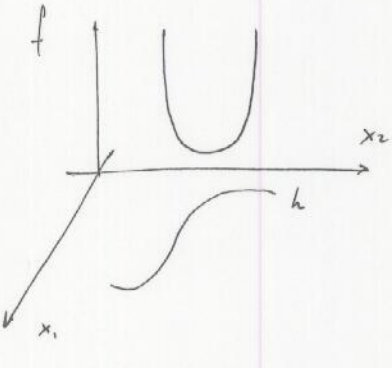
$$g : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad h : \mathbb{R}^n \longrightarrow \mathbb{R}^\ell$$

- NECESSARY CONDITIONS:

KKT [KARUSH - KUHN - TUCKER]

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$h : \mathbb{R}^n \longrightarrow \mathbb{R}^\ell$$
$$g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

f = the function to minimize



NECESSARY CONDITIONS:

IF $f \in C^1(\mathbb{R}^n)$ and $x^* \in \mathbb{R}^n$ IS A LOCAL MINIMA of the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ SUBJECT TO the CONSTRAINTS, then:

$\exists \; \mu \in \mathbb{R}^m , \; \lambda \in \mathbb{R}^e$ such that :

$$\frac{\partial f}{\partial x}\left(x^*\right) + \sum_{i=1}^{m} \mu_i \frac{\partial g_i}{\partial x}\left(x^*\right) + \sum_{j=1}^{e} \lambda_j \frac{\partial h_j}{\partial x}\left(x^*\right) = 0 \qquad [1]$$

$\lambda \longrightarrow$ LAGRANGE MULTIPLIERS

$$g\left(x^*\right) \leq 0 , \quad h\left(x^*\right) = 0$$

$$\mu_i \geq 0 , \quad \mu_i g_i\left(x^*\right) = 0 \quad i = 1, \dots, m$$

$$g(x) = \begin{bmatrix} g_1(x) \\ \vdots \\ g_m(x) \end{bmatrix} , \quad h(x) = \begin{bmatrix} h_1(x) \\ \vdots \\ h_e(x) \end{bmatrix}$$

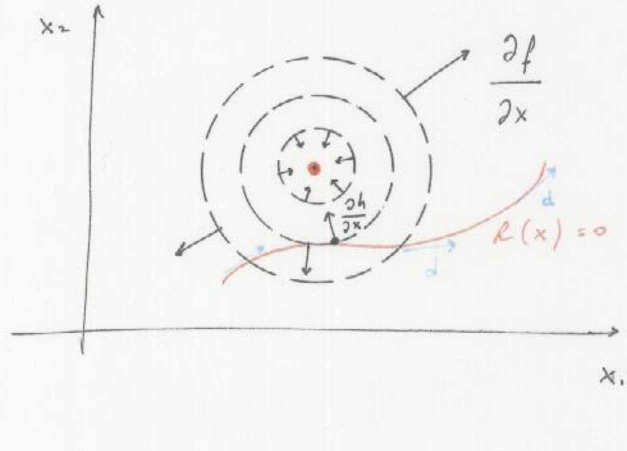
■

[1] is the CORE of EVERYTHING!

• NOTE: Example in 2D

$n = 2$

• = the DERIVATIVE is ϕ
FLAT POINT



→ you modify the formula in:

$$[2] \quad (*) \begin{cases} \dfrac{\partial f}{\partial x}(x^*) \cdot d = 0 \\ \forall d \text{ (possible direction) COMPATIBLE with the the CONSTRAINT} \end{cases}$$

that is in this situation:

$\frac{\partial f}{\partial x}$ is the gradient of the cost function, this gradient in the point x^*, must be orthogonal to all the possible directions compatible with the constraint

NOTE: the CONDITION on d is the same of [1] except the component of μ,

$$\frac{\partial f}{\partial x}(x^*) + \sum_{i=1}^{\ell} \lambda_i \frac{\partial h}{\partial x}(x^*) = 0$$

so to OPTIMIZE you function you go to the center of the CIRCLE

. the set $\mathcal{D}$ of the ADMISSIBLE CONSTRAINTS IS DEFINED AS FOLLOWS:

$$[3] \quad \mathcal{D}(x) = \left\{ d \in \mathbb{R}^n : \underbrace{d \cdot \frac{\partial h_i(x)}{\partial x} = 0}_{d \text{ it's ortogonal to the } h}, \ i = 1 \ldots \underbrace{\ell}_{\text{dimension of } h} \right\}$$

→ if you do a 1° ORDER Taylor expansion around x you found the condition here

[3] is the matematica equivalent of [2]

$$(*)^1 \quad \frac{\partial f}{\partial x}(x^*) \cdot d = 0 \quad \forall d \in \mathcal{D} = \left\{ d : d \cdot \frac{\partial h_i(x^*)}{\partial x} = 0, \ i = 1..e \right\}$$

$\hookrightarrow$ this is equivalent to an another condition:

$$(*)^2 \quad \frac{\partial f}{\partial x}(x^*) \in \underbrace{\operatorname{span}\left\{ \frac{\partial h_1}{\partial x}(x^*), \dots, \frac{\partial h_e}{\partial x}(x^*) \right\}}_{[\mathcal{D}(x^*)]^\perp}$$

$\hookrightarrow$ that is equivalent to [1]

. The gradient is a linear combination of the constraint

; 2 SPACE are ORTOGONAL

- Dimostration by contradiction: (of equivalence : $(*)^1 = (*)^2$)

Let's assume $\frac{\partial f}{\partial x}(x^*) \notin \operatorname{span}\left\{ \frac{\partial h_1}{\partial x}(x^*) \dots \frac{\partial h_e}{\partial x}(x^*) \right\}$

then we can write $\frac{\partial f}{\partial x}(x^*) = \underbrace{d}_{\in \mathcal{D}(x^*)} + \underbrace{\sum_{i=1}^{e} \lambda_i \frac{\partial h_i}{\partial x}(x^*)}_{\in [\mathcal{D}(x^*)]^\perp}$

$$\frac{\partial f}{\partial x}(x^*) \cdot d = \underbrace{d \cdot d}_{\neq \phi} + \underbrace{\left[\cancel{\sum_{i=1}^{e} \lambda_i \frac{\partial h_i}{\partial x}(x^*)} \right] \cdot d}_{= \phi}$$

$\hookrightarrow$ contradiction!

because can't be: $\frac{\partial f}{\partial x}(x^*) \cdot d = 0$

$\Rightarrow$ so the 2 are equivalent!

the 'CONSTRAINTS' are the DYNAMIC of the SYSTEM !
quindi. the OPTIMIZATION in X^* WILL BE REPLACE WITH
TRAJECTORIES ! ⟹ the Long term IDEA is to have the calculus of variations to operate not on VECTORS but on FUNCTIONS ! in the space of FUNCTIONS !! (TRAJECTORIES)

if i wanna go from 1 to 2 with the minimum lenght curve you have to analyze all the possible curves and prove that the straight line is the min. traj.