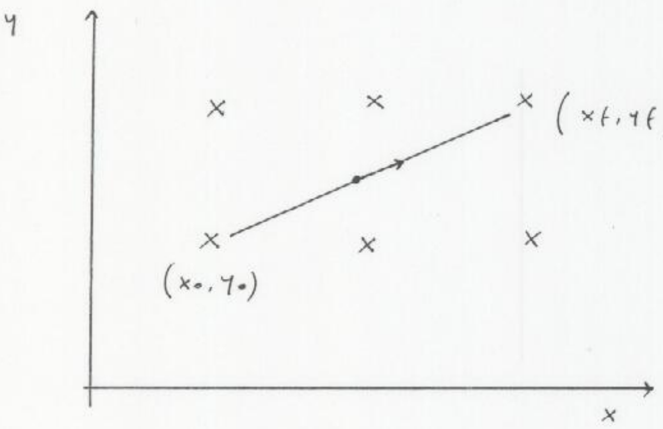
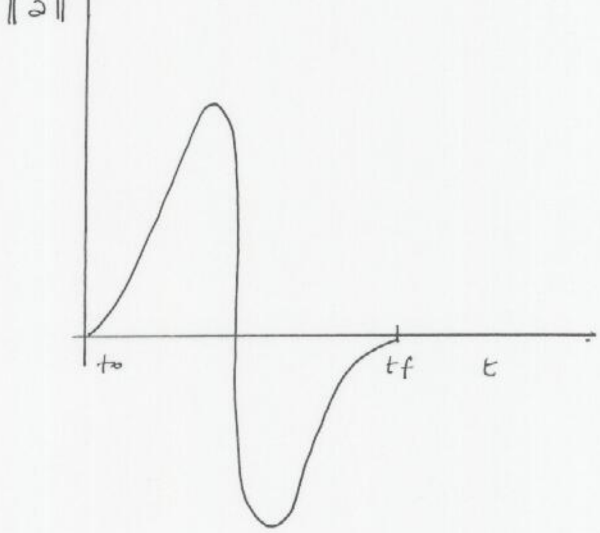
Biorobotics: Lesson 2

- Pietro MORASSO ⟶ MIT: with the "BRACCIO di FERRO" or "MIT-MANUS", measure Trajectory of a point to point movement



1) LINEAR TRAJECTORY



- $x(\cdot), y(\cdot)$ is such that

$$\min_{x(\cdot)\, y(\cdot)} \int_{t_0}^{t_f} \left[x^{(3)}(\tau) \right]^2 + \left[y^{(3)}(\tau) \right]^2 d\tau \qquad \text{s.t.} \longrightarrow$$

MIN - JERK

$$\longrightarrow \begin{cases} x(t_0) = x_0, & y(t_0) = y_0 \\ x(t_f) = x_f, & y(t_f) = y_f \end{cases}$$

- today we solve the min of JERK; we are optimizing over all possible Trajectories. (it's a kinematic feature, a way to make it smooth and more conservative).

- See 2 papers on the website : ✓ MORASSO
  ✓ HOGAN

$\longrightarrow$ "OLD" PAPERS

- The tool we use is : $\boxed{\text{CALCULUS of VARIATIONS}}$

if you say that

$$X^*(t) = \arg \min_{x(\cdot)} \int_{t_0}^{t_f} L\left(x(\tau), \dot{x}(\tau), \tau\right) d\tau$$

is the traj. that minimize

$$\begin{cases} x(t_0) = x_0 \\ x(t_f) = x_f \end{cases}$$

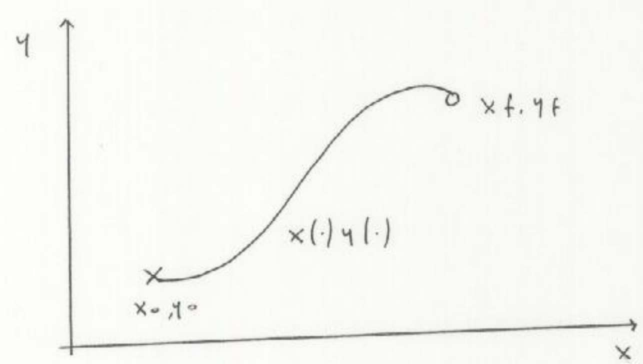
$$X : [t_0, t_f] \longrightarrow \mathbb{R}^n$$

IF $\quad x^* : [t_0, t_1] \longrightarrow \mathbb{R}^n$ is a maximum or a minimum (i.e. extremum) then :

then: $$\begin{cases} \frac{\partial L}{\partial x}\left(x^*(t), \dot{x}^*(t), t\right) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\left(\dot{x}(t), \dot{x}^*(t), t\right) = 0 & \forall\, t \in [t_0, t_f] \\ x^*(t_0) = x_0 \\ x^*(t_f) = x_f \end{cases}$$

- Some Applications to have an idea of how strong it is:

Example: If I want to go from a point 1 to another, with a minimum lenght



$$X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\arg\min_{x(\cdot)\,y(\cdot)} \int_{t_0}^{t_f} \underbrace{\left[\dot{x}^2(\tau) + \dot{y}^2(\tau)\right]^{1/2}}_{\text{lenght of a general path}} d\tau$$

$$\underbrace{\begin{matrix} x(t_0) = x_0 \\ x(t_f) = x_f \end{matrix}}_{\text{Initial cond.}} \qquad \underbrace{\begin{matrix} y(t_0) = y_0 \\ y(t_f) = y_f \end{matrix}}_{\text{final cond}}$$

$$L\left(x(t), \dot{x}(t), y(t), \dot{y}(t), \tau\right)$$

more precisely: $$L\left(x(t), y(t), \dot{x}(t), \dot{y}(t), t\right) = \left[\dot{x}^2(t) + \dot{y}^2(t)\right]^{1/2}$$

$$\frac{\partial L}{\partial x} = \begin{bmatrix} \dfrac{\partial L}{\partial x} \\[2ex] \dfrac{\partial L}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 \\[2ex] 0 \end{bmatrix}$$

$$\frac{\partial L}{\partial \dot{x}} = \begin{bmatrix} \dfrac{\partial L}{\partial \dot{x}} \\[2ex] \dfrac{\partial L}{\partial \dot{y}} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2} \cdot \dfrac{1}{\left[\dot{x}^2(t) + \dot{y}^2(t)\right]^{\frac{1}{2}}} \, 2\dot{x} \\[3ex] \dfrac{1}{2} \, \dfrac{1}{\left[\dot{x}^2(t) + \dot{y}^2(t)\right]^{1/2}} \, 2\dot{y} \end{bmatrix}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \begin{bmatrix} \dfrac{1}{2} \cdot \left(-\dfrac{1}{2}\right) \cdot \dfrac{1}{\left[\dot{x}^2(t) + \dot{y}^2(t)\right.} \, 2\dot{x} \cdot \ddot{x} \; 2\dot{x} + \dfrac{1}{2} \, \dfrac{1}{\left[\dot{x}^2(t) + \dot{y}^2(t)\right]^{\frac{1}{2}}} \, 2\ddot{x} \\[6ex] \; \end{bmatrix}$$

$$\frac{1}{2}\left(-\frac{1}{2}\right) \frac{1}{\left[\dot{x}^2(t) + \dot{y}^2(t)\right]^{\frac{3}{2}}} \left(2\dot{x}\ddot{x} + 2\dot{y}\ddot{y}\right)\left(2\dot{x}\right) + \frac{1}{2} \cdot \frac{2\ddot{x}}{\left[\dot{x}^2(t) + \dot{y}^2(t)\right]^{\frac{1}{2}}} =$$

$$= \frac{-\dot{x}\left(\dot{x}\ddot{x} + \dot{y}\ddot{y}\right) + \ddot{x}\left(\dot{x}^2 + \dot{y}^2\right)}{\left[\dot{x}^2 + \dot{y}^2\right]^{3/2}} = \frac{\cancel{-\dot{x}^2\ddot{x}} - \dot{x}\,\dot{y}\ddot{y} + \cancel{\ddot{x}\dot{x}^2} + \ddot{x}\dot{y}^2}{\left[\dot{x}^2 + \dot{y}^2\right]^{3/2}}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \begin{bmatrix} \dfrac{\dot{y}\left(\dot{y}\ddot{x} - \dot{x}\ddot{y}\right)}{\left[\dot{x}^2 + \dot{y}^2\right]^{3/2}} \\[3ex] \dfrac{\dot{x}\left(\dot{x}\ddot{y} - \dot{y}\ddot{x}\right)}{\left[\dot{x}^2 + \dot{y}^2\right]^{3/2}} \end{bmatrix} = \begin{bmatrix} 0 \\[3ex] 0 \end{bmatrix}$$

$$\dot{y}\left(\dot{y}\ddot{x} - \dot{x}\ddot{y}\right) = 0$$

$\hookrightarrow$ Excluding the case $\dot{x} = \dot{y} = 0$ (still point that doesn't extend the curve) $\longrightarrow$ must be:

Assume that:

$$\dot{y}\ddot{x} - \dot{x}\ddot{y} = 0 = \frac{d}{dt}\left(\frac{\dot{x}}{\dot{y}}\right)\cdot \dot{y}^2 = \frac{d}{dt}\left(\frac{\frac{dx}{dt}}{\frac{dy}{dt}}\right)\cdot \dot{y}^2 =$$

$$= \frac{d}{dt}\left(\frac{dx}{dy}\right)\dot{y}^2 = 0$$

$\downarrow$

it's done! IF we assume that $\dot{y}^2 \neq 0$

$$\Rightarrow \boxed{\frac{dx}{dy} = \text{const}}$$

$\hookrightarrow$ so the curve that connects the two points it's a LINE!

• LEAST ACTION PRINCIPLE

$\longrightarrow$ Mechanical systems when moving from a point to another minimize the ACTION.

Given a physical system we can associate to the system a mathematical quantity called the "ACTION" wich is an attribute of the system (it's a property of the Dynamical System).

The "ACTION" is a FUNCTIONAL, i.e. an operator which takes as INPUT TRAJECTORIES of the system $q(t)$ $t \in [t_0, t_f]$ and associates REAL NUMBERS to the trajectory.

A way to represent the ACTION is $\mathcal{S}$ consists in integrating a function $L(q(\cdot), \dot{q}(\cdot), t)$ of the system generalized coordinates $q(\cdot)$ and generalized velocities $\dot{q}(t)$:

$$\mathcal{S}[q(\cdot)] = \int_{t_0}^{t_f} L(q(\tau), \dot{q}(\tau), \tau)\, d\tau$$

$\hookrightarrow$ the OPERATOR S defines you a SCALAR. When the ACTION of a GIVEN DYNAMICAL SYSTEM CAN BE DESCRIBED IN THIS way, the FUNCTION $L(\cdot,\cdot,\cdot)$ IS CALLED the SYSTEM LAGRANGIAN.

- Theorem (HAMILTON'S PRINCIPLE (or LEAST ACTION PRINCIPLE)): the True evolution of a mechanical system described by the generilize coordinates $q(t) \in \mathbb{R}^n$ between two points q_0 and q_f is a stationary point[1] of the ACTION FUNCTIONAL:

$$\mathcal{S}[q(\cdot)] = \int_{t_0}^{t_f} L(q(\tau), \dot{q}(\tau), \tau)\, d\tau$$

$\hookrightarrow$ if you have a real system, to go to q_0 to q_f the function moves from a minimum to a max.

$\mathcal{S}$ has to satisfy:

$$\begin{cases} \dfrac{\partial L}{\partial q} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}} = 0 \\ q(t_0) = q_0, \quad q(t_f) = q_f \end{cases}$$

[1] A POINT WHERE LOCAL MODIFICATION OF THE TRAJECTORY $q(\cdot)$ CORRESPONDS TO ZERO VARIATION

- Example: if the system is a kinematic chain with n-degree of freedom, then,

$$L(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q)$$

→ the way to compute the DYNAMIC of the system

you get to usual:

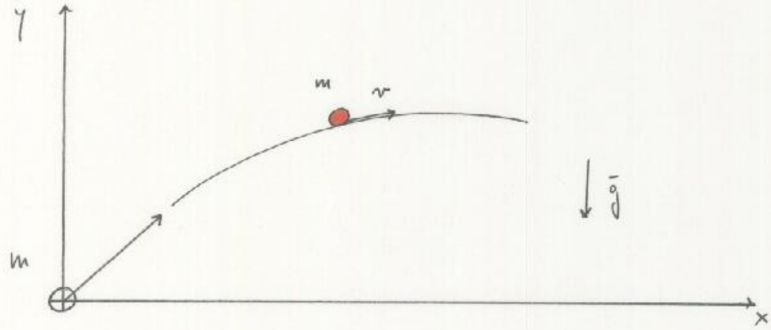
$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = 0$$

→ EULER-LAGRANGE Equation it's a way to compute the DYNAMIC EQUATION. Imposing the necessary conditions for a stationary point of the associated action: $\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$ we exactly get the system dynamics.

- NOETHER THEOREM → if you look at the system, the Lagrange, and if you consider the Lagrange moved by the q the system doesn't change (if you have symmetry in that function)

→ Look at it!

- Example: consider the problem of throwing a mass and describing the dynamic



$$q(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$L = \underbrace{\frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right)}_{T} - \underbrace{mgy}_{V}$$

if you apply NEWTON's LAW WE ALREADY KNOW that the SYSTEM DYNAMICS ARE REGULATED BY:

$$\begin{cases} m\ddot{x} = 0 \\ m\ddot{y} = -mg \end{cases} \longrightarrow \text{which gives the SYSTEM EVOLUTION from:}$$

$$x(0) = x_0, \quad y(0) = y_0 \qquad \text{i. c.}$$
$$\dot{x}(0) = \dot{x}_0, \quad \dot{y}(0) = \dot{y}_0$$

FROM the LEAST ACTION PRINCIPLE,
the ACTION of the TRAJECTORY q is:

$$S\left[q(\cdot)\right] = \int_{t_0}^{t_f} \left[\frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) - mgy\right] dt$$

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -m\ddot{x} = 0$$

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = -mg - m\ddot{y} = 0$$

Using this theorem you can write:

$$\begin{cases} m\ddot{x} = 0 \\ m\ddot{y} = -mg \end{cases} \qquad x(0) = x_0, \quad y(0) = y_0, \quad x(t_f) = x_f \quad y(t_f) = y_f$$

→ instead specify the velocity you specify the final position

THEREFORE DYNAMICS of the SYSTEM ARE THE SAME BUT BOUNDARY CONDITIONS ARE DIFFERENT BUT IN BOTH CASES A UNIQUE SOLUTION of the DYNAMIC EQUATIONS CAN BE DETERMINED.

Generalization of CALCULUS of VARIATIONS:

$$x^*(t) = \arg\min_{x(\cdot)} \int_{t_0}^{tf} L\left(x(\tau), \dot{x}(\tau), \dots, x^{n}(\tau), \tau\right) d\tau$$

$$\text{s.t.} \quad x(t_0) = x_0, \quad x(tf) = xf \quad \dots \quad x^{(n-1)}(t_0) = x_0^{(n-1)}, \quad x^{(n-1)}(tf) = xf^{(n-1)}$$

$$\begin{cases} \dfrac{\partial L}{\partial x} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}} + \underbrace{\dots}_{(-1)^2 \frac{d^2}{dt^2}\frac{\partial L}{\partial x^{(2)}}} + (-1)^n \dfrac{d^n}{dt^n}\dfrac{\partial L}{\partial x^{(n)}} = 0 \\ x(t_0) = x_0, \dots, x^{(n-1)}(t_0) = x_0^{(n-1)} \\ x(tf) = xf, \dots, x^{(n-1)}(tf) = xf^{(n-1)} \end{cases}$$

→ the sign changes because the integral:

$$\int_{t_0}^{tf} f\dot{g} + \int_{t_0}^{tf} \dot{f} g = f \cdot g \Big|_{t_0}^{tf}$$

Example of the minimal JERK TRAJECTORY:

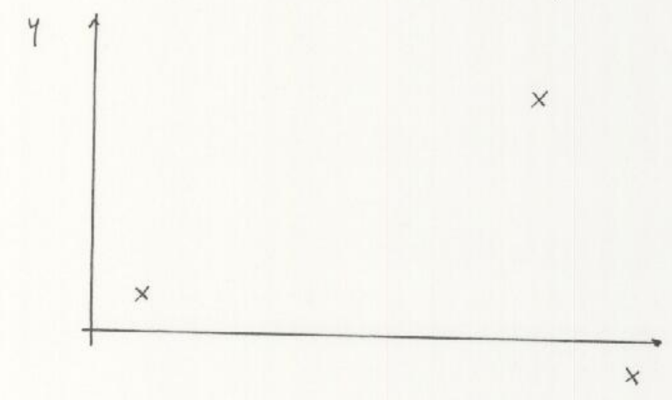
$$\min_{x(\cdot)\,y(\cdot)} \int_{t_0}^{tf} \left[x^{(3)^2}(\tau) + y^{(3)^2}(\tau) \right] d\tau$$

→ we need to min this function, with boundary condition:

$x(t_0) = x_0 \quad x(tf) = xf$

$y(t_0) = y_0 \quad y(tf) = yf$

→ the INTEGRAL is a LINEAR FUNCTION and can be decomposed in two parts : $\min\limits_{xy} \int x^{(3)2} + \min\limits_{xy} \int y^{(3)2}$



$\hookrightarrow \min\limits_{x} \int x^{(3)2} + \min\limits_{y} \int y^{(3)2}$

i can basically solve one problem because the other is the same :

$$\min_{x(\cdot)} \frac{1}{2}\int_0^T \left[x^{(3)}\right]^2(\tau)\, d\tau$$

$x(0) = x_0 \ , \ x(T) = x_T$

we assume that the others initial conditions are ϕ:

$\dot{x}(0) = 0 \ , \ \dot{x}(T) = 0$

$\ddot{x}(0) = 0 \ , \ \ddot{x}(T) = 0$

the LAGRANGIAN :

$$L\left(x(t), x^{(1)}(t), x^{(2)}(t), x^{(3)}(t), t\right) = \left[x^{(3)}(t)\right]^2$$

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \dot{x}} = 0$$

$$\frac{\partial L}{\partial x^{(2)}} = 0$$

$$\frac{\partial L}{\partial x^{(3)}} = \cancel{2} x^{(3)}(t) \cdot \frac{1}{\cancel{2}}$$

$$(-1)^3 \frac{d^3}{dt^3} \frac{\partial L}{\partial x^{(3)}} = \underbrace{(-1)\, x^{(6)}(t) = 0}_{\text{LINEAR DIFFERENTIAL EQUATION}} \Longrightarrow x(t) = a_0 + a_1 t + \dots + a_5 t^5$$

SOLUTION: POLINOMIAL 5^{th} ORDER

$$x(0) = x_0 = a_0$$

$$\dot{x}(0) = a_1 = 0$$

$$\ddot{x}(0) = 2a_2 = 0$$

$$a_0 = x_0, \quad a_1 = a_2 = 0$$

$$x(T) = x_0 + a_3 T^3 + a_4 T^4 + a_5 T^5 = x_T$$

$$\dot{x}(T) = 3a_3 T^2 + 4a_4 T^3 + 5a_5 T^4 = 0$$

$$\ddot{x}(T) = 6a_3 T + 12 a_4 T^2 + 20 a_5 T^3 = 0$$

$$\longrightarrow 3a_3 T^2 = -4a_4 T^3 - 5a_5 T^4 \Longrightarrow a_3 T^2 = -\frac{4}{3} a_4 T^3 - \frac{5}{3} a_5 T^4 \ *$$

$$-8a_4 T^2 - 10 a_5 T^3 + 12 a_4 T^2 + 20 a_5 T^3 = 0$$

$$10 a_5 T^3 = -4 a_4 T^2 \Longrightarrow \boxed{a_5 = -\frac{2}{5} \frac{1}{T} a_4}$$

$$(x_0 - x_T) = -a_3 T^3 - a_4 T^4 - a_5 T^5$$

$$* = -\frac{4}{3} a_4 T^3 - \frac{\cancel{5}}{3} \left(-\frac{2}{\cancel{5}} \frac{1}{T} a_4\right) T^4$$

$$a_3 T^2 = -\frac{4}{3} a_4 T^3 + \frac{2}{3} T^3 a_4$$

$$\boxed{a_3 T^2 = -\frac{2}{3} a_4 T^3}$$

$$(x - x_T) = +\frac{2}{3} a_4 T^4 - a_4 T^4 + \frac{2}{5}\frac{1}{T} a_4 T^{\cancel{5}\,4} = \frac{10-15+6}{15} a_4 T^4$$

$$a_4 = \frac{15\,(x_0 - x_T)}{T^5}$$

$$a_3 = -\frac{2}{3} T \cdot \frac{15}{T^4}(x - x_T) = -\frac{10\,(x_0 - x_T)}{T^3}$$

$$a_5 = -\frac{2}{5}\cdot\frac{1}{T}\,\frac{15\,(x_0 - x_T)}{T^4} = -\frac{6\,(x_0 - x_T)}{T^4}$$

$$\boxed{x(t) = x_0 - \frac{10\,(x_0 - x_T)}{T^3} t^3 + \frac{15\,(x_0 - x_T)}{T^4} t^4 - \frac{6\,(x_0 - x_T)}{T^5} t^5}$$

MINIMUN JERK TRAJECTORY