- Biorobotics: Lesson N° 7

- Summary:

OPTIMAL CONTROL PROBLEM:

$$\min_{u(\cdot)} \int_{t_0}^{t_f} g\left(x(\tau), u(\tau), \tau\right) d\tau + \underbrace{h\left(x(t_f), t_f\right)}_{\text{in this case, fixed final state, it's costant}}$$

$$\text{s.t.} \quad \dot{x}(t) = f\left(x(t), u(t)\right)$$

$$x(t_0) = x_0, \quad \left(\dot{x}(t_f) = x_f\right)$$

$$\begin{cases} \dot{x}^*(t) = \dfrac{\partial \mathcal{H}}{\partial p}\left(x^*(t), u^*(t), p(t), t\right) \\ \dot{p}(t) = -\dfrac{\partial \mathcal{H}}{\partial x}\left(x^*(t), u^*(t), p(t), t\right) \\ 0 = \dfrac{\partial \mathcal{H}}{\partial u}\left(x^*(t), u^*(t), p(t), t\right) \end{cases}$$

$$\left(\frac{\partial h}{\partial x}\left(x^*(t_f), t_f\right) - p^*(t_f) = 0\right)$$

$$\min_{u} \int_{t_0}^{t_f} \left[\frac{1}{2} x^T(\tau) Q x(\tau) + \frac{1}{2} u^T(\tau) R u(t)\right] d\tau + \frac{1}{2} x^T(t_f) Q_f x(t_f)$$

$$\text{s.t.} \quad \begin{cases} \dot{x} = Ax + Bu \\ x(t_0) = x_0, \quad \left(x(t_f) = x_f\right) \end{cases}$$

$$u^*(t) = -R^{-1}B^T p(t)$$

$$\begin{cases} \dot{x}^*(t) = A x^*(t) - BR^{-1}B^T p^*(t) \\ \dot{p}^*(t) = -Q x(t) - A^T p^*(t) \end{cases} \Longrightarrow (*)$$

$Q_f x^*(t_f) = p(t_f)$ BOUNDARY CONDITION $(*)$

$K(t)\, x(t) = p(t)$ KALMAN

$$(*) \quad \underbrace{\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix}}_{\dot{\mathrm{x}}(t)} = \underbrace{\begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix}}_{\mathrm{x}(t)}$$

$$\dot{\mathrm{x}}(t) = \mathcal{A}\, \mathrm{x}(t)$$
$$\mathrm{x}(t_0) = \mathrm{x}_0$$

if $\mathcal{A}$ is time invariant, which is the solution of this dynamica eq. ?

$$\mathrm{x}(t) = e^{\mathcal{A}(t-t_0)}\, \mathrm{x}(t_0)$$

$$e^{\mathcal{A}} = \sum_{i=0}^{\infty} \frac{\mathcal{A}^i}{i!}$$

In general, if $\mathcal{A}$ is time INVARIANT, the SOLUTION is:

$$x(t) = \Phi(t, t_0)\, x(t_0) \xrightarrow[t \to t_0]{t_f \to t} (**)$$

where $\Phi(t, t_0)$ satisfies the following

$$\begin{cases} \dot{\Phi}(t, t_0) = \mathcal{A}(t)\, \Phi(t, t_0) \\ \Phi(t_0, t_0) = I \end{cases}$$

For the computations, we start from the CONDITION $(*)$

$$(**) \quad x(t_f) = \Phi(t_f, t)\, x(t)$$

$$\underbrace{\begin{bmatrix} x(t_f) \\ p(t_f) \end{bmatrix}}_{x(t_f)} = \underbrace{\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}}_{\Phi(t_f, t)} \underbrace{\begin{bmatrix} x(t) \\ p(t) \end{bmatrix}}_{x(t)}$$

$$x(t_f) = \Phi_{11}(t_f, t)\, x(t) + \Phi_{12}(t_f, t)\, p(t)$$

$$p(t_f) = \Phi_{21}(t_f, t)\, x(t) + \Phi_{22}(t_f, t)\, p(t) = Q_f\, x(t_f)$$

$$\boxed{\begin{aligned} Q_f x^*(t_f) &= Q_f \Phi_{11}(t_f, t)\, x^*(t) + Q_f \Phi_{12}(t_f, t)\, p^*(t) = \\ &= \Phi_{21}(t_f, t)\, x^*(t) + \Phi_{22}(t_f, t)\, p^*(t) \end{aligned}}$$

$\longrightarrow$ Eq. that is similar to : $\boxed{K(t)\, x^*(t) = p^*(t)}$

$$\underbrace{\left[Q_f \Phi_{11}(t_f, t) - \Phi_{21}(t_f, t) \right]}_{\text{IF this part is INVERTIBLE :}} x^*(t) = \left[\Phi_{22} - Q_f \Phi_{12} \right] p^*(t)$$

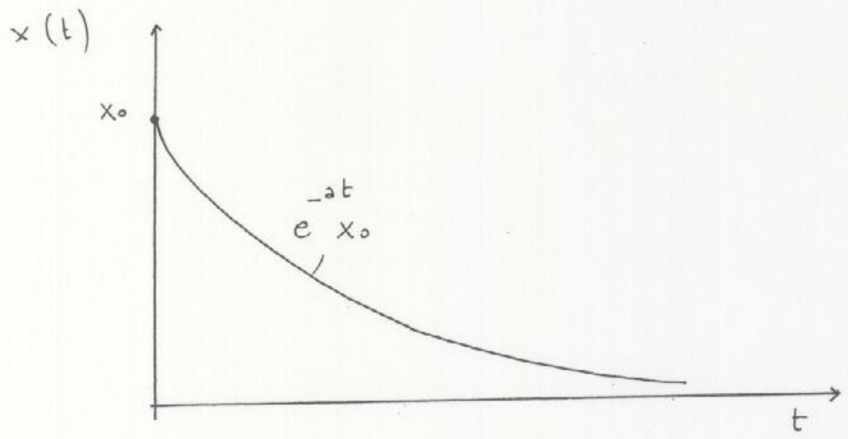
$$x^*(t) = \underbrace{\left[Q_f \Phi_{11} - \Phi_{21} \right]^{-1} \left[\Phi_{22} - Q_f \Phi_{12} \right]}_{K(t)} p^*(t)$$

$\longrightarrow$ this proves that the relation between the state X and p is scaled from the Gain K

- Example:

$$\dot{x}(t) = \underbrace{a}_{A}\, x(t) + u(t), \qquad x(t) \in \mathbb{R}$$

$$\min_u \int_0^T \underbrace{\frac{1}{4}}_{\frac{1}{2} \cdot \underbrace{\frac{1}{2}}_{R}} u^2(\tau)\, d\tau + \frac{1}{2} \underbrace{H}_{Q_f} x^2(T)$$



it could be a capacitor plus the voltage that i applied; another possibility could be a chemical solution with a reactor and in a fixed time interval i want to cancel the quantity $\dot{x}$.

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} a & -2 \\ 0 & -a \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix}$$

Q is ϕ because we don't have the cost on the state

↳ Computing the Exponential of the matrix $\mathcal{A}$:

$$\mathcal{A} = \begin{bmatrix} a & -2 \\ 0 & -a \end{bmatrix}, \quad \mathcal{A}^2 = \begin{bmatrix} a & -2 \\ 0 & -a \end{bmatrix} \begin{bmatrix} a & -2 \\ 0 & -a \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix}$$

$$\mathcal{A}^3 = \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix} \begin{bmatrix} a & -2 \\ 0 & -a \end{bmatrix} = \begin{bmatrix} a^3 & -2a^2 \\ 0 & -a^3 \end{bmatrix}$$

$$\mathcal{A}^4 = \begin{bmatrix} a^3 & -2a^2 \\ 0 & -a^3 \end{bmatrix} \begin{bmatrix} a & -2 \\ 0 & -a \end{bmatrix} = \begin{bmatrix} a^4 & 0 \\ 0 & a^4 \end{bmatrix}$$

$$\mathcal{A}^k = \begin{bmatrix} a^k & \frac{1}{a}\left(-a^k + (-1)^k a^k\right) \\ & (-1)^k a^k \end{bmatrix} \longrightarrow e^{\mathcal{A}} = \left[\begin{array}{c|c} e^{a} & -\frac{1}{a}e^{a} + \frac{1}{a}e^{-a} \\ \hline 0 & e^{-a} \end{array}\right]$$

$$\mathcal{A}^5 = \begin{bmatrix} a^4 & 0 \\ 0 & a^4 \end{bmatrix} \begin{bmatrix} a & -2 \\ 0 & -a \end{bmatrix} = \begin{bmatrix} a^5 & -2a^4 \\ 0 & -a^5 \end{bmatrix}$$

$$\mathcal{A} = \sum_{k=0}^{\infty} \frac{\mathcal{A}^k}{k!} = \begin{bmatrix} \overbrace{\sum_k \frac{a^k}{k!}}^{e^{a}} & \overbrace{-\frac{1}{a}\sum_k \frac{a^k}{k!}}^{-\frac{1}{a}e^{a}} + \overbrace{\frac{1}{a}\sum_k \frac{(-a)^k}{k!}}^{\frac{1}{a}e^{-a}} \\ 0 & \underbrace{\sum_k \frac{(-a)^k}{k!}}_{e^{-a}} \end{bmatrix}$$

$$\Phi(t_f, t) = e^{\mathcal{A}(t_f - t)}$$

$$e^{\mathcal{A}(tf-t)} = \left[\begin{array}{c|c} e^{a(tf-t)} & -\frac{1}{a}e^{a(tf-t)} + \frac{1}{a}e^{-a(tf-t)} \\ \hline 0 & e^{-a(tf-t)} \end{array}\right]$$

$$\Phi(tf,t) = e^{\mathcal{A}(tf-t)} = \left[\begin{array}{c|c} \overset{\Phi_{11}}{e^{a(tf-t)}} & \overset{\Phi_{12}}{-\frac{1}{a}e^{(tf-t)} + \frac{1}{a}e^{(tf-t)}} \\ \hline \underset{\Phi_{21}}{0} & \underset{\Phi_{22}}{e^{-a(tf-t)}} \end{array}\right]$$

$$k(t) = \left[H e^{a(tf-t)} \right]^{-1} \left[e^{-a(tf-t)} - \frac{H}{a} \cdot e^{a(tf-t)} + \frac{H}{a} e^{-a(tf-t)} \right]$$

$$u^*(t) = -2He^{-a(tf-t)}\left[-\frac{H}{a} e^{a(tf-t)} + \frac{H}{a} e^{-a(tf-t)} \right] x(t)$$

if i increase H, somehow i have to have a bigger control.

- The DUALITY betwenen OPEN LOOP and CLOSED LOOP:

a POSSIBLE SOLUTION:

$$u(t),\ t \in [t_0, tf]$$

OR:

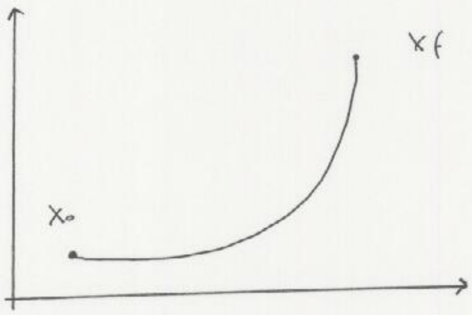
$$u^*(t) = -R^{-1} B^T k(t)\, x(t) \longrightarrow$$

→ it's very important because sometimes the FEEDBACK of the SYSTEM it's too SLOW. This is VERY IMPORTANT BECAUSE in some SITUATIONS (LIKE PARKOUR) in which the CONTROL HAVE TO BE FASTER (FEEDBACK LOOPS ≃ 130 ms). → Which is the TRADE OFF BETWEEN the TWO CONTROL STRATEGIES?

- CONSIDER the PARTICULAR PROBLEM:

$$\begin{cases} \dot{x} = A x(t) + B u(t) \\ x(t_0) = x_0, \quad x(t_f) = x_f \end{cases}$$

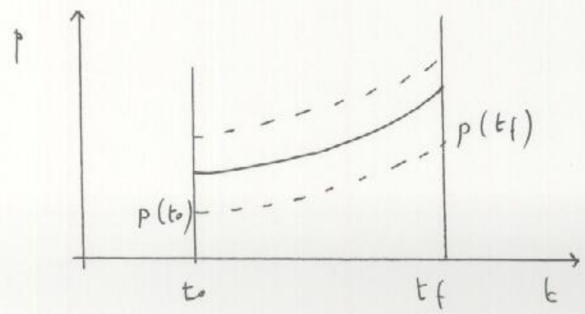
$$\frac{1}{2} \int_{t_0}^{t_f} u^T(\tau) R\, u(\tau)\, d\tau$$



, want to MINIMIZE the EFFORT from x_0 to x_f

$$\begin{cases} \dot{x} = A x(t) - B R^{-1} B^T p(t) \\ \dot{p} = -A^T p(t) \longrightarrow p(t) = e^{A^T (t_f - t)} p(t_f) \end{cases}$$

$$\dot{x} = A x(t) - \underbrace{B R^{-1} B^T e^{A^T (t_f - t)} p(t_f)}_{-\omega(t)}$$



if I know the final CONDITION, I know the behaviour. I can either chose one in the middle and everything it's the same

$$\begin{cases} \dot{x} = Ax + Bu, \quad x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau)\, d\tau \\ x(t_0) = x_0 \end{cases}$$

$\longrightarrow$ EVOLUTION of the STATE

$$x(t) = e^{A(t-t_0)} x_0 \underbrace{- \int_{t_0}^{t} e^{A(t-t_0)} B R^{-1} B^T e^{A^T(tf-t)}}_{G(t_0,t)} p(tf)\, d\tau$$

$G(t_0, t)$ GRAMIAN MATRIX

$\longrightarrow$ it's invertable; if the system is CONTROLLABLE, this is INVERTABLE.
G is INVERTIBLE IF AND ONLY IF the SYSTEM (A, B) is CONTROLLABLE

$$x(t) = e^{A(t-t_0)} x_0 - G(t_0, t)\, p(tf)$$

$$x(tf) = e^{A(tf-t_0)} x_0 - G(t_0, tf)\, p(tf) = xf$$

$$p(tf) = -G^{-1}(t_0, tf)\left[xf - e^{A(tf-t_0)} x_0\right]$$

$$u^*(t) = + R^{-1} B^T G^{-1}(t_0, tf)\left[xf - e^{A(tf-t_0)} x_0\right]$$

$\longrightarrow$ SOLUTION that is a sort of OPEN LOOP CONTROL because the state doesn't depend on t

- $$G(t_0, t) = \int_{t_0}^{t} e^{A(t-\tau)} B R^{-1} B^T \underbrace{e^{A^T(tf-t)}}_{e^{A^T(\tau-\tau)} \cdot e^{A^T(tf-t)}} d\tau$$

$$G(t_0,t) = \underbrace{\int_{t_0}^{t} e^{A(t-\tau)} B R^{-1} B^T e^{A^T(t-\tau)} d\tau}_{\hat{G}(t_0,t)} \; e^{A^T(t_f-t)}$$

$$u^*(t) = R^{-1} B^T e^{A^T(t_f-t)} \underbrace{G^{-1}(t_0,t_f)\left[x_f - e^{A(t_f-t_0)} x_0\right]}_{-p(t_f)}$$

$$\left[e^{A}\right]^{-1} = e^{-A}$$

- Example: to compute the Gradient explicity [pay ATTENTION !]

MIN JERK ~~example~~ PROBLEM

$$\frac{1}{2}\int_0^T \left[\frac{d^3x}{dt^3}(\tau)\right]^2 d\tau \qquad \text{s.t.} \qquad \begin{array}{l} x(0)=x_0\,,\ \dot{x}(0)=0\,,\ \ddot{x}(0)=0 \\ x(T)=x_f\,,\ \dot{x}(T)=0\,,\ \ddot{x}(T)=0 \end{array}$$

IDEA: write as a LINEAR SYSTEM

$$\boxed{\frac{d^3x}{dt^3} = u} \qquad \text{DYNAMICS of the SYSTEM}$$

we have to define the STATE as:

$$\dot{X} = AX + Bu$$

and in order to define this we have to define X:

$$X = \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \dot{X} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dddot{x} \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ u \end{bmatrix} \Longrightarrow$$

$$\Rightarrow \quad \dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{A \text{ NILPOTENT MATRIX}} x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u$$

we can prove that

$$A^3 = 0$$