Nori
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- Biorobotics: Lesson N° 8

- Examples: we'll see one analitical example and then one numerical

- Example 1: MINIMUN JERK

$$\min_u \int_{t_0}^{t_f} g\left(x(\tau), u(\tau), \tau\right) d\tau \qquad \text{s.t.} \qquad \begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ x(t_0) = x_0, \ x(t_f) = x_f \end{cases}$$

$$\begin{cases} 0 = \dfrac{\partial \mathcal{H}}{\partial u}\left(x(t), p(t), u(t), t\right) \\ \dot{x} = \dfrac{\partial \mathcal{H}}{\partial p}\left(x(t), p(t), u(t), t\right) \\ \dot{p} = -\dfrac{\partial \mathcal{H}}{\partial x}\left(x(t), p(t), u(t), t\right) \end{cases}$$

$$\mathcal{H}\left(x(t), p(t), u(t), t\right) = g\left(x(t), u(t), t\right) + p^T(t)\, f\left(x(t), u(t)\right) \qquad \text{HAMILTONIAN}$$

the 1° Example is ANALITICAL; if we write:

$$\begin{cases} \dot{x} = Ax + Bu & \text{LINEAR DYNAMICS} \\ x(t_0) = x_0, \ x(t_f) = x_f \end{cases}$$

$$g\left(x(t), u(t), t\right) = \frac{1}{2} u^T(t)\, R\, u(t) \qquad \text{QUADRATIC COST FUNCTION}$$

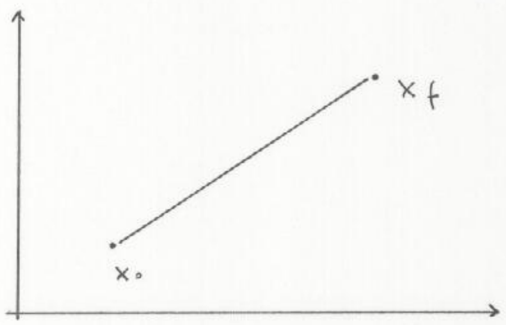
$$u^*(t) = R^{-1} B^T e^{A^T(t_f - t)}\, G^{-1}(t_0, t_f) \left[x(t_f) - e^{A(t_f - t)} x(t_0)\right]$$

$$G(t_0, t_f) = \int_{t_0}^{t_f} e^{A(t_f-\tau)} B R^{-1} B^T e^{A^T(t_f-\tau)} \, d\tau$$

suppose that I want to solve the problem:

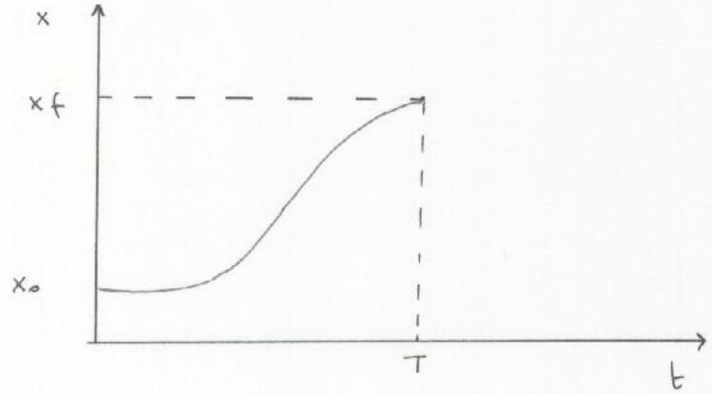
$$\min_x \int_0^T \frac{1}{2} \underbrace{\left[\dddot{x}(\tau) \right]^2}_{u^2} d\tau \qquad \text{s.t.} \quad x(0) = x_0, \quad x(T) = x_f$$

this is a good model for a PLANAR REACHING MOVEMENT: we follow a trajectory that minimize the $\dddot{x}$

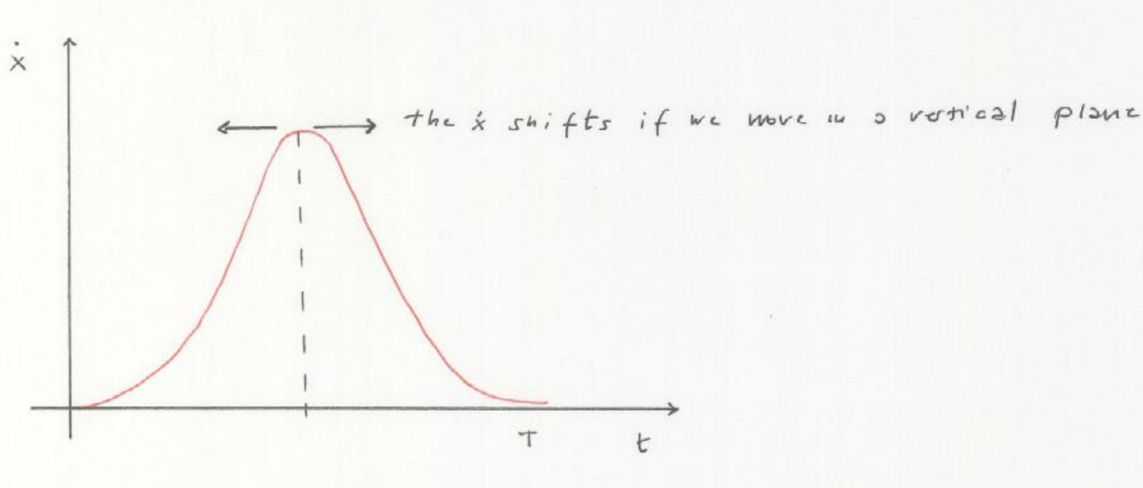


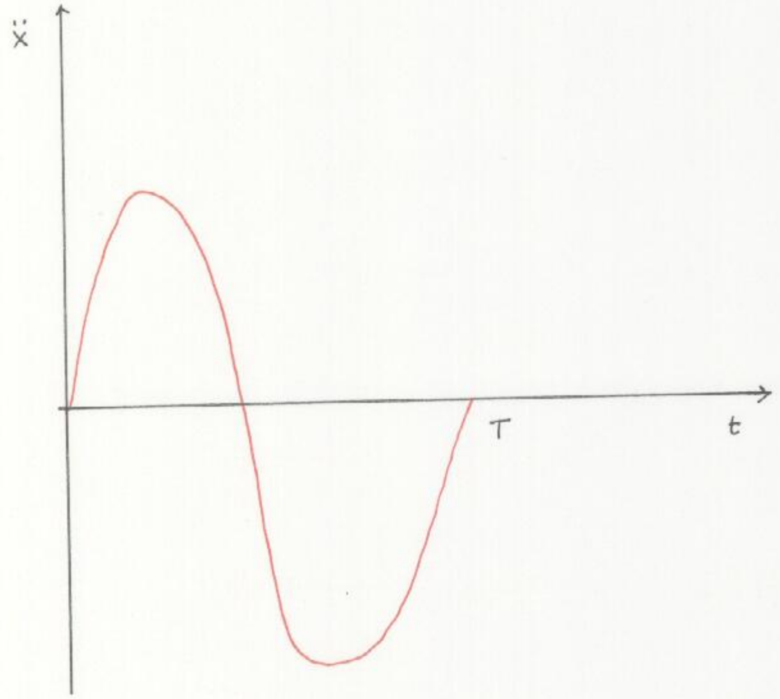
$$x(t) = x_0 + (x_f - x_0)\left[10 \frac{(T-t)^3}{T^3} - 15 \frac{(T-t)^4}{T^4} + 6 \frac{(T-t)^5}{T^5} \right]$$

if you plot $x(t)$:



→ Human movements always have this kind of behaviour [see MORASSO'S PAPER ON THE SITE!]





the STATE that I define is:

$$X(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix}, \quad \dot{X}(t) = \begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \\ \dddot{x}(t) \end{bmatrix}, \quad u(t) = \dddot{x}(t)$$

$$\dot{X}(t) = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{A} X(t) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u(t)$$

$$e^{At} = \sum_{i=0}^{\infty} \frac{A^i t^i}{i!} = I + At + \frac{A^2 t^2}{2!}$$

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

NIHILPOTENT MATRIX

→ Every time we have a DIAGONAL, if we do the power of the matrix, the DIAGONAL WILL SHIFT and then the SINGLE ONE ○ will become a ZERO ○*

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & & 0 & 1 \\ 0 & & & 0 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$ Exponential matrix

PROPERTY that comes from the linearity of the exponential

$$e^{A^T} = \left[e^{A} \right]^T$$

$$e^{A^T(T-t)} = \begin{bmatrix} 1 & 0 & 0 \\ T-t & 1 & 0 \\ \frac{(T-t)^2}{2} & T-t & 1 \end{bmatrix}$$

$$G(0,T) = \int_0^T \begin{bmatrix} 1 & T-\tau & \frac{(T-\tau)^2}{2} \\ 0 & 1 & T-\tau \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} 1 \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ T-\tau & 1 & 0 \\ \frac{(T-\tau)^2}{2} & T-\tau & 1 \end{bmatrix}$$

$$= \int_0^T \begin{bmatrix} \frac{(T-\tau)^2}{2} \\ T-\tau \\ 1 \end{bmatrix} \begin{bmatrix} \frac{(T-\tau)^2}{2} & (T-\tau) & 1 \end{bmatrix} d\tau$$

$$= \int_0^T \begin{bmatrix} \frac{(T-\tau)^4}{4} & \frac{(T-\tau)^3}{2} & \frac{(T-\tau)^2}{2} \\ \frac{(T-\tau)^3}{2} & (T-\tau)^2 & T-\tau \\ \frac{(T-\tau)^2}{2} & T-\tau & 1 \end{bmatrix} d\tau$$

$$= -\begin{bmatrix} \frac{(T-\tau)^5}{20} & \frac{(T-\tau)^4}{8} & \frac{(T-\tau)^3}{6} \\ \frac{(T-\tau)^4}{8} & \frac{(T-\tau)^3}{3} & \frac{(T-\tau)^2}{2} \\ \frac{(T-\tau)^3}{6} & \frac{(T-\tau)^2}{2} & -\tau \end{bmatrix}_0^T =$$

$$= \begin{bmatrix} \frac{T^5}{20} & \frac{T^4}{8} & \frac{T^3}{6} \\ \frac{T^4}{8} & \frac{T^3}{3} & \frac{T^2}{2} \\ \frac{T^3}{6} & \frac{T^2}{2} & T \end{bmatrix}$$

$$u^*(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ T-t & 1 & 0 \\ \frac{(T-t)^2}{2} & T-t & 1 \end{bmatrix} \overbrace{\begin{bmatrix} \frac{720}{T^5} & -\frac{360}{T^4} & \frac{60}{T^3} \\ -\frac{360}{T^4} & \frac{192}{T^3} & -\frac{36}{T^2} \\ \frac{60}{T^3} & -\frac{36}{T^2} & \frac{9}{T} \end{bmatrix}}^{G^{-1}(0,T) \text{ COMPUTED with SYMBOLIC TOOLS: OR MAPLE}} \left[x(t_f) - \underbrace{\begin{bmatrix} 1 & T & \frac{T^2}{2} \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}}_{e^{AT}} x(t_0) \right]$$

$$x(t_0) = \begin{bmatrix} x(t_0) \\ \dot{x}(t_0) \\ \ddot{x}(t_0) \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \\ 0 \end{bmatrix}, \quad x(t_f) = \begin{bmatrix} x(t_f) \\ \dot{x}(t_f) \\ \ddot{x}(t_f) \end{bmatrix} = \begin{bmatrix} x_f \\ 0 \\ 0 \end{bmatrix}$$

$$u^*(t) = \begin{bmatrix} \frac{(T-t)^2}{2} & T-t & 1 \end{bmatrix} G^{-1}(0,T) \left[\begin{bmatrix} x_f \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} x_0 \\ 0 \\ 0 \end{bmatrix} \right]$$

$$u^*(t) = \begin{bmatrix} \frac{(T-t)^2}{2} & T-t & 1 \end{bmatrix} \begin{bmatrix} \frac{720}{T^5}(x_f - x_0) \\ -\frac{360}{T^4}(x_f - x_0) \\ \frac{60}{T^3}(x_f - x_0) \end{bmatrix}$$

$$u^*(t) = \left[\frac{\cancel{720}^{360}}{T^5} \frac{(T-t)^2}{\cancel{2}} - \frac{360}{T^4}(T-t) + \frac{60}{T^3} \right] (x_f - x_0) = \dddot{x}(t)$$

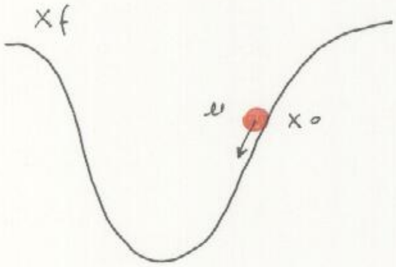
- MATLAB

→ SYMBOLIC TOOLBOX of MATLAB: abstract variables, here T it's a symbol

```
>> syms T real
>> G = [T^5/20, T^4/8, ..]
>> pretty (G)   % scrive G in modo visualizzabile
```

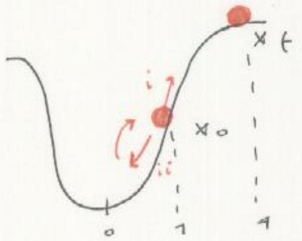
```
>> inv (G)
>> syms foo real
>> syms pippo real
>> f = pippo ^ 2
>> diff (f, pippo)
```

Suppose we have a 1 dimensional mass moving in a potential FIELD (a sort of valley) and we have an initial configuration, I have a FORCE u and I want to drive the mass from x_0 to x_f using the minimum Energy



$$\min \int_0^T u^2(\tau)\, d\tau$$

in this case:



is not so EASY

(i) - should i push directly from x_0 to x_f?
(ii) - should i go down, take Energy and then go to x_f?

iF you have sufficient TIME you can go down and take energy (solution ii), but if i don't have sufficient T i go directly toward x_f.

$T = 2 \longrightarrow$ the mass is push directly upward (i)

$T = 12 \longrightarrow$ you let the ball fall down at the beginning a little bit (ii)

- the NECESSARY CONDITIONS are the following:

$$\frac{\partial \mathcal{H}}{\partial u}\Big(x(t), u(t), p(t), t\Big) = 0 \Longrightarrow \underbrace{u^*\big(x(t), p(t), t\big)}$$ STATICAL CONDITION

$$\begin{cases} \dot{x} = \dfrac{\partial \mathcal{H}}{\partial p}\Big(x(t), u(t), p(t), t\Big) \\ \dot{p} = -\dfrac{\partial \mathcal{H}}{\partial x}\Big(x(t), u(t), p(t), t\Big) \end{cases}$$

$x(t_0) = x_0, \quad x(t_f) = x_f$

$\longrightarrow$ BOUNDARY CONDITIONS PARTICULAR CASE

the procedure it's always the same:

1) SOLVE •
2) SOSTITUTE in ⊙ and ▲

$$\underbrace{\begin{cases} \dot{x} = \dfrac{\partial \mathcal{H}}{\partial p}\Big(x(t), u^*\big(x(t), p(t), t\big), p(t), t\Big) \\ \dot{p} = -\dfrac{\partial \mathcal{H}}{\partial x}\Big(x(t), u^*\big(x(t), p(t), t\big), p(t), t\Big) \end{cases}}$$

$$\dot{X}(t) = F\big(X(t), t\big)$$

$$X(t) = \begin{bmatrix} x(t) \\ p(t) \end{bmatrix}$$

$$\begin{cases} \dot{x}(t) = F\left(x(t), t\right) \\ x(t_0) = \begin{bmatrix} x_0 \\ p_0 \end{bmatrix} \end{cases} \Longrightarrow x(t) \quad t \in [t_0, t_f]$$

$$\dot{x}(t) \cong \frac{x\left(t + \Delta t\right) - x(t)}{\Delta t} = F\left(x(t), t\right)$$

$$x\left(t + \Delta t\right) = x(t) + \Delta t \, F\left(x(t), t\right) \longrightarrow$$ compute the solution every Δt

$$\begin{cases} \dot{x}(t) = F\left(x(t), t\right) \\ x(t_0) = \begin{bmatrix} x_0 \\ p_0 \end{bmatrix} \end{cases} \Longrightarrow x\left(t; p_0\right) \quad t \in [t_0, t_f]$$

$$x\left(t; p_0\right) = \begin{bmatrix} x(t, p_0) \\ p(t, p_0) \end{bmatrix}$$

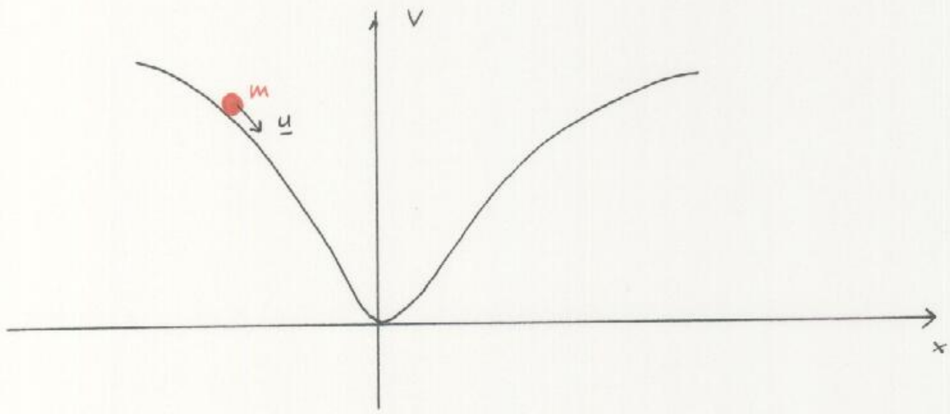
$\longrightarrow$ $\boxed{\text{SHOOTING METHOD}}$

If the solution it is optimal it must exist a p_0 that hits the target
we are trying to hit the target, every time I try a different p_0 and then i use every time a random force to do this

$$\min_{p_0} \left\| x\left(t_f; p_0\right) - x_f \right\|$$

$\longrightarrow$ instead of solving an optimization problem I check if I hit the TARGET
we try to understand LOCALLY what happens
NUMERICAL PROBLEM

POTENTIAL EQUATION :



$$V(x) = \left[\operatorname{atan}(x)\right]^2$$ POTENTIAL FUNCTION

LAGRANGE APPROACH: $L = K - V$

$$K = \frac{1}{2} m \dot{x}^2$$

$$L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - \left[\operatorname{atan}(x)\right]^2$$ LAGRANGE EQ.

$\underline{u}$ = force applied to the mass

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = u$$

$$m\ddot{x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}, \quad \frac{\partial L}{\partial x} = -\frac{2\operatorname{atan}(x)}{x^2+1}$$

$$\boxed{m\ddot{x} + \frac{2\operatorname{atan}(x)}{x^2+1} = u} \Longrightarrow m\ddot{x} + b\dot{x} + \frac{2\operatorname{atan}(x)}{x^2+1} = u$$

considering NON CONSERVATIVE FORCES

the CONTROL PROBLEM:

$$\min_u \int_0^T \frac{1}{2} u^2(\tau)\, d\tau$$

$$x(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}, \quad \dot{x}(t) = \begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix} = \begin{bmatrix} \dot{x} \\ -\frac{b}{m}\dot{x}(t) - \frac{2}{m}\frac{\operatorname{atan}(x)}{x^2+1} + \frac{u}{m} \end{bmatrix}$$

$$\text{s.t.} \begin{cases} \dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_1(t) \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{b}{m}x_2 - \frac{2}{m}\frac{\operatorname{atan}(x_1)}{1+x_1^2} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \\ x(0) = \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix}, \quad x(T) = \begin{bmatrix} x_f \\ \dot{x}_f \end{bmatrix} \end{cases}$$

ARBITRARY CONDITIONS

to compute the solution, we define the HAMILTONIAN:

$$\mathcal{H}\left(x(t), u(t), p(t), t\right) = \frac{1}{2}u^2(t) + \underbrace{p^T(t)}_{\left[p_1(t)\ p_2(t)\right]} \begin{bmatrix} x_2 \\ -\frac{b}{m}x_2 - \frac{2}{m}\frac{\operatorname{atan}(x_1)}{1+x_1^2} + \frac{u}{m} \end{bmatrix}$$

$$\frac{\partial \mathcal{H}}{\partial u} = 0 = u(t) + \frac{p_2(t)}{m}$$

$$\boxed{u^*\left(x(t), p(t), t\right) = -\frac{p_2(t)}{m}}$$

1° CONDITION

the 2° comes from:

$$\frac{\partial \mathcal{H}}{\partial p} = \dot{x} = \begin{bmatrix} x_2 \\ -\frac{b}{m}x_2 - \frac{2}{m}\frac{\operatorname{atan}(x_1)}{1+x_1^2} + \frac{1}{m}\left(-\frac{p_2(t)}{m}\right) \end{bmatrix}$$

$$-\dot{p} = \frac{\partial \mathcal{H}}{\partial x} = \begin{bmatrix} -\frac{2}{m} p_2(t) & \frac{1 + 2x_1 \operatorname{atan}(x_1)}{(1+x_1^2)^2} \\ p_1(t) - p_2(t)\frac{b}{m} & \end{bmatrix}$$

$$\frac{d}{dx_1}\left[\frac{\operatorname{atan}(x_1)}{1+x_1^2}\right] = \frac{\frac{1}{\cancel{1+x_1^2}}\left(\cancel{1+x_1^2}\right) - 2x_1 \operatorname{atan}(x_1)}{(1+x_1^2)^2} =$$

$$= \frac{1 - 2x_1 \operatorname{atan}(x_1)}{(1+x_1^2)^2}$$

$\hookrightarrow$ we have only $\dot{x}, \dot{p}$ as unknown, I don't have u because I replace u with

- MATLAB:

I have to minimize the cost function by changing the initial condition

error = [x (end) - xf ; dx (end) - dxf].^2 $\Rightarrow$ WE OPTIMIZE THIS ERROR

the dynamical eq. describe the evolution of the dynamical state :

dx(1) = x2
dx(2) =
dx(3) =
dx(4) =

ODE 45 $\longrightarrow$ INTEGRATE DIFFERENTIAL EQ.

fsolve