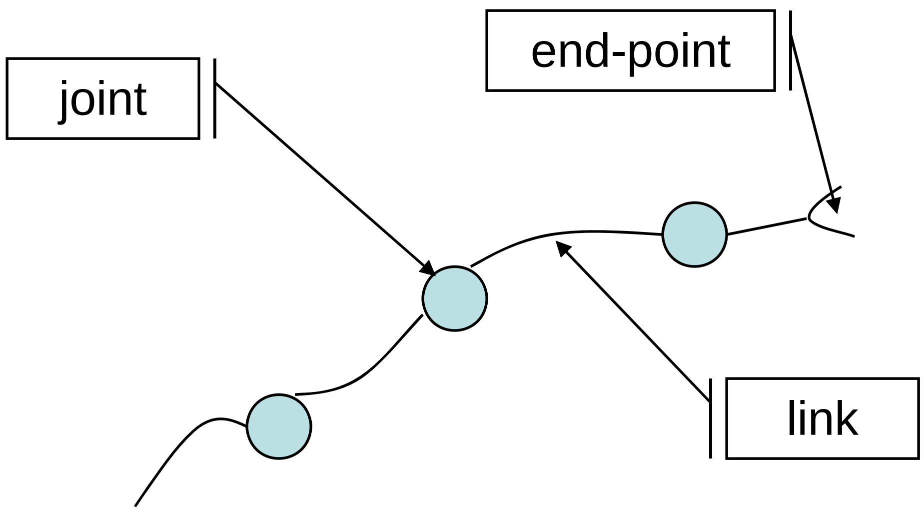
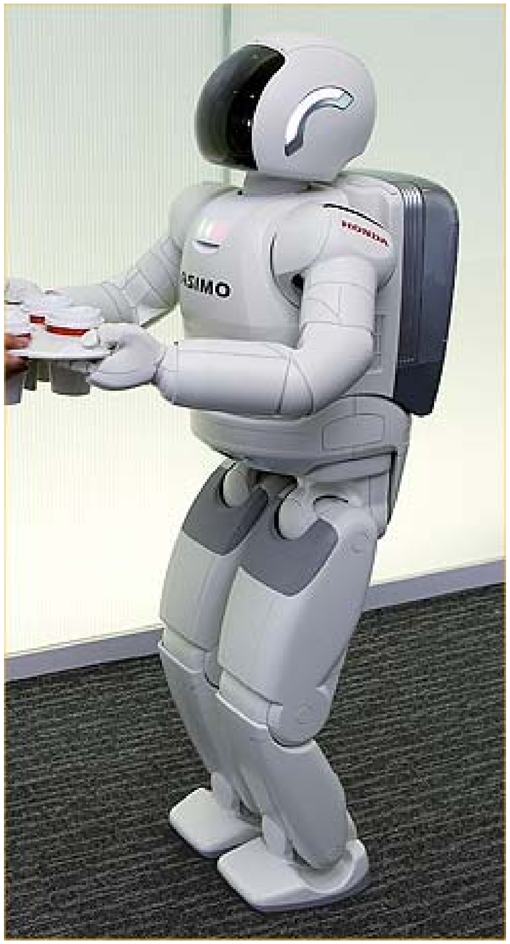
Mechanical systems

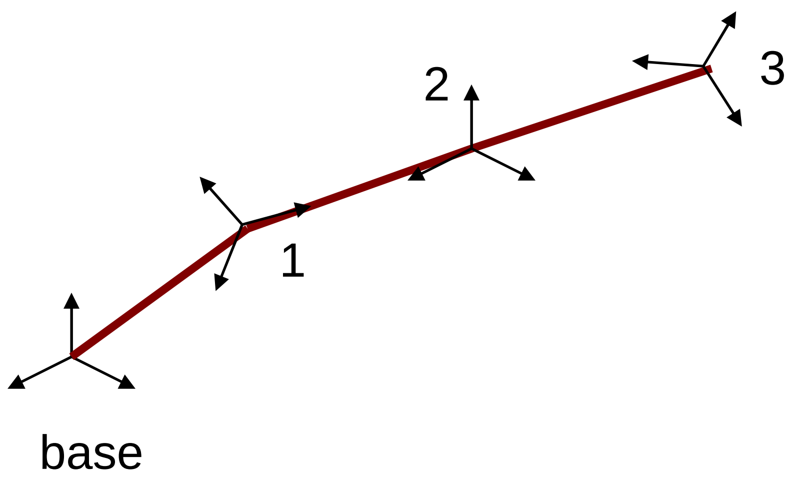
- Things we'd like to model with the help of some trivial physics





How to describe things mathematically

- One reference frame per link
  - Not needed for now…



Studying what?

	No forces	Forces
No motion	Styling	Static
Motion	Kinematics	Dynamics

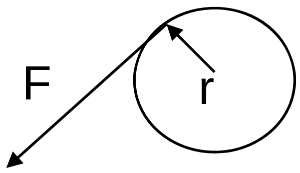
Notation

$$F = \frac{d}{dt}(mv) = m\ddot{x}$$

Since links are physical objects with mass

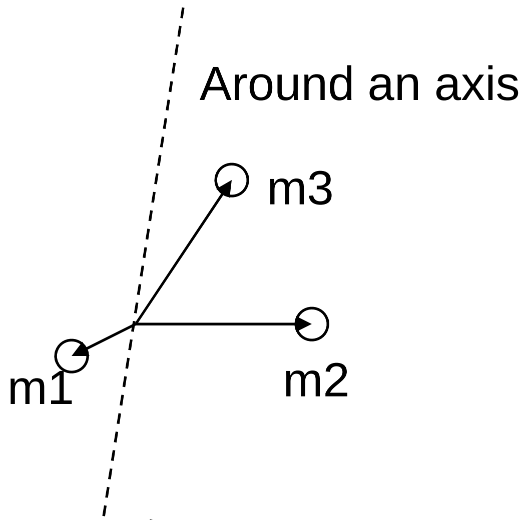
$$\tau = J\ddot{\vartheta}$$

J = moment of inertia

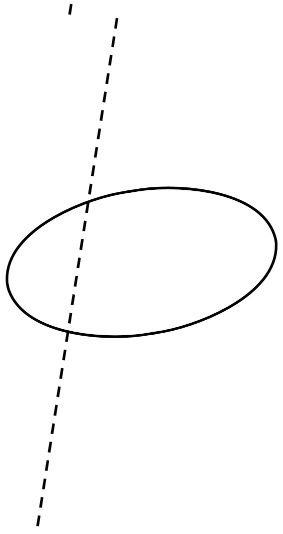


$$\tau = F \times r$$

Moment of inertia



$$J = \sum_{i=1}^{N} m_i r_i^2$$



density

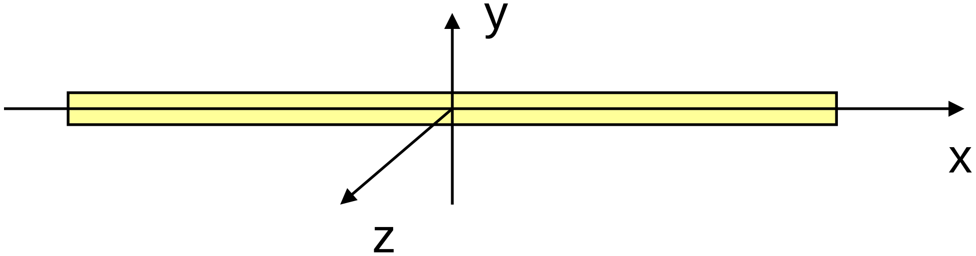
$$J = \int_{volume} \rho r^2 dV$$

Parallel axis theorem

$$J = J_c + Mr^2$$

Through the center of gravity

Example



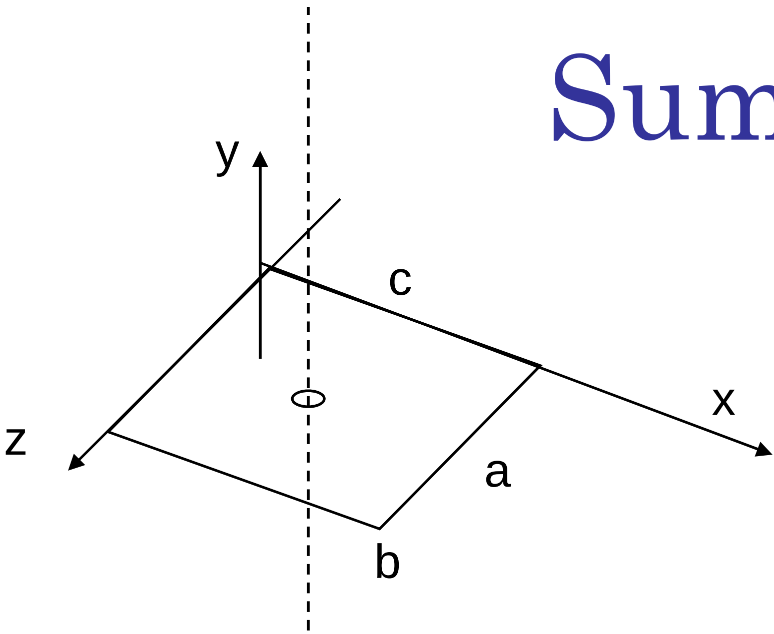
$$Mass = M\,,\ \rho = M/l$$

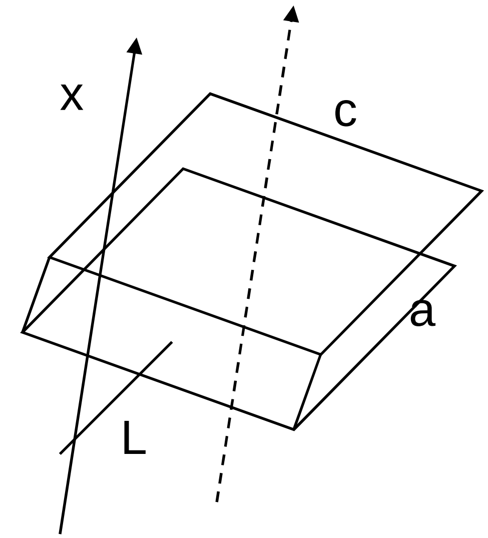
$$J_x = 0$$

$$J_y = \rho\int r^2 dV = \rho\int_{-l/2}^{l/2} x^2 dx = \rho\frac{1}{3}x^3\Big|_{-l/2}^{l/2} = \frac{Ml^2}{12}$$

$$J_{y=-l/2} = \frac{Ml^2}{12} + M\frac{l^2}{4} = M\frac{l^2}{3}$$

Sum of J





$$J_x = \frac{M}{12}(a^2 + b^2)$$

$$J_y = \frac{M}{12}(a^2 + c^2)$$

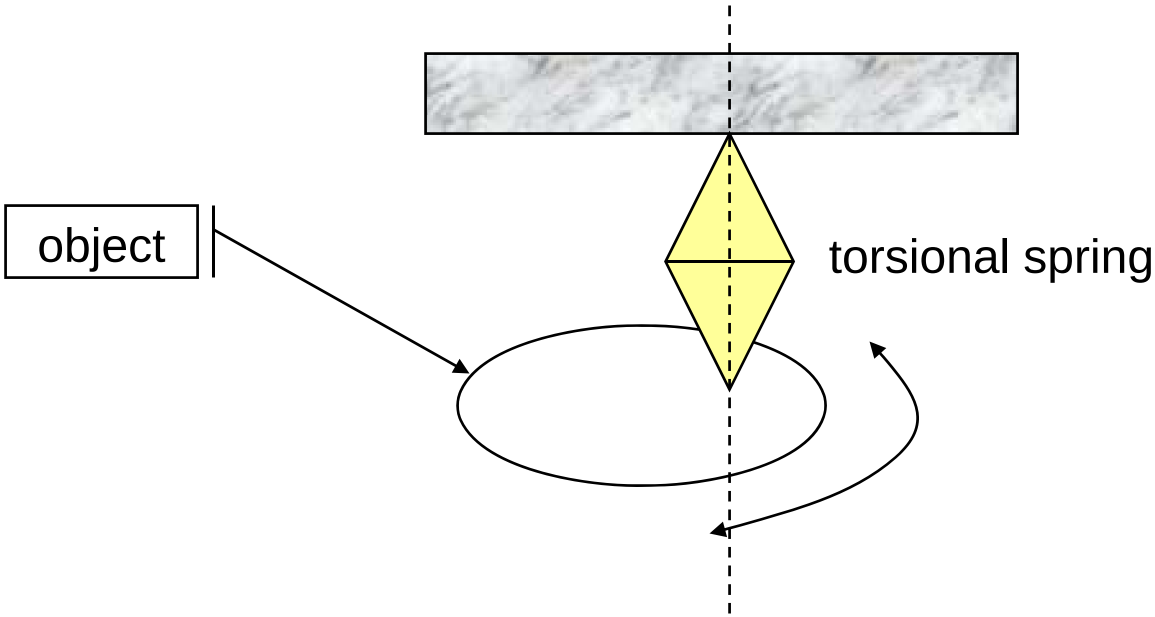
$$J_z = \frac{M}{12}(b^2 + c^2)$$

e.g. $\rightarrow J_{top-x} = \frac{M_{top}}{12}(a^2 + c^2) + M_{top}(\frac{a}{2} + L)^2$

$$\downarrow$$

$$J_{hand-x} = J_{top-x} + J_{side-x} + J_{bottom-x}$$

Experimental estimation of J

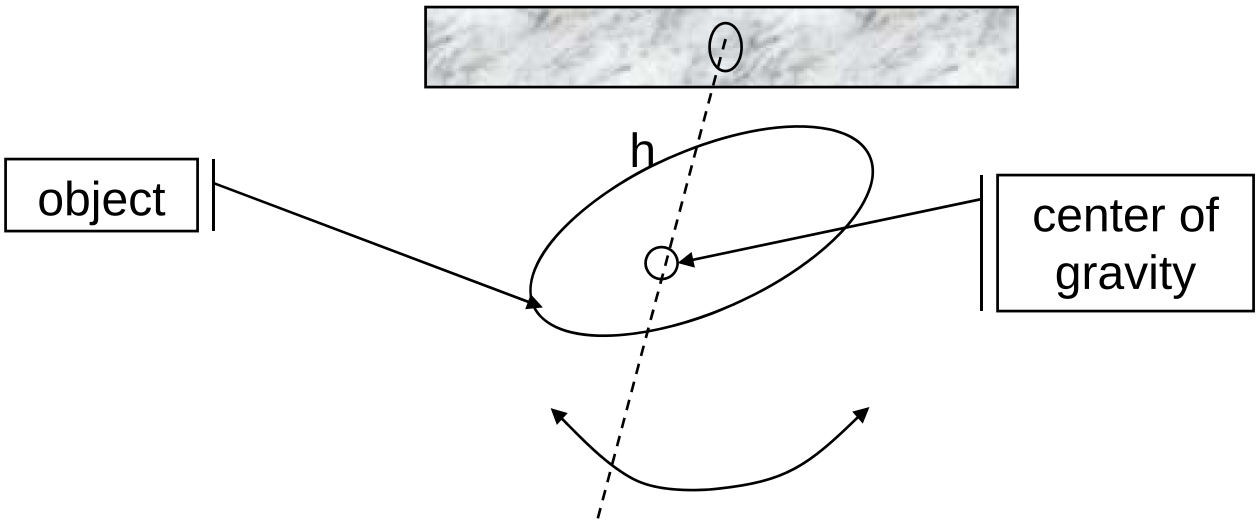


Use a photodiode and a computer to measure the frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{K}{J}}$$

Requires calibration from known *J*

Experimental estimation of J



$$f \approx \frac{1}{2\pi}\sqrt{\frac{Mgh}{J}}$$

Work and power

$$E = const \qquad \textbf{\textit{if}} \qquad \sum F_{ext} = 0$$

$$W = \int_{s1}^{s2} F ds \qquad W = \Delta E, E = \text{energy}$$

$$K = \frac{1}{2} m v^2$$

kinetic energy

$$P = \frac{dW}{dt}$$ Power $\rightarrow$ $$P = Fv$$

Rotational case

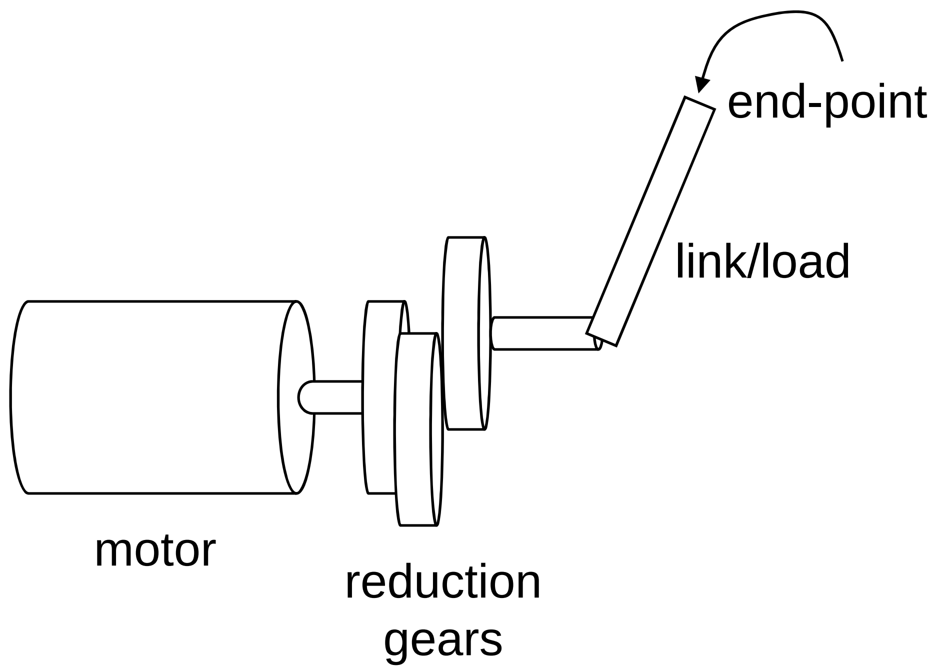
$$E = const \qquad \textbf{\textit{if}} \qquad \sum \tau_{ext} = 0$$

$$W = \int_{\vartheta 1}^{\vartheta 2} \tau d\vartheta \qquad W = \Delta E, E = \text{energy}$$

$$K = \frac{1}{2} J \omega^2$$ **kinetic energy**

$$P = \frac{dW}{dt}$$ Power $\rightarrow$ $$P = \tau\omega$$

As I mentioned, we'd like to model a single joint



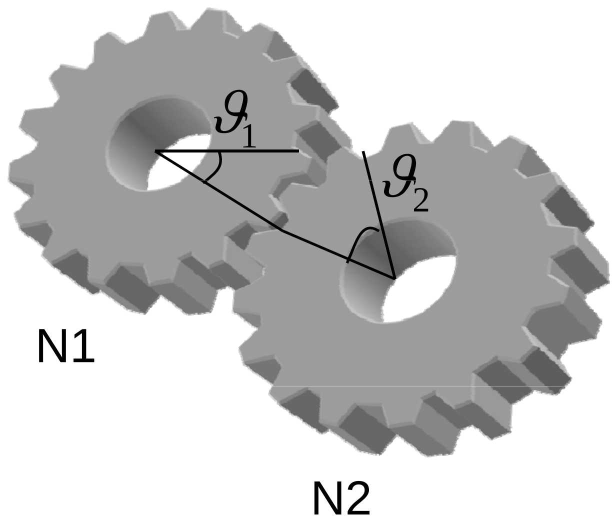
Motor

- Let's imagine for now that it is something that generates a given torque

Mechanical transmission

- Gears
- Belts
- Lead screws
- Cables
- Cams
- etc.

Gears



- Distance traveled is the same:

$$r_1\vartheta_1 = r_2\vartheta_2$$

- Because the size of teeth is the same:

$$\frac{N_1}{r_1} = \frac{N_2}{r_2}$$

Furthermore…

$$r_1\vartheta_1 = r_2\vartheta_2$$

$$\frac{N_1}{r_1} = \frac{N_2}{r_2}$$

- No loss of energy $\tau_1\vartheta_1 = \tau_2\vartheta_2$

Combining...

$$\frac{N_1}{N_2} = \frac{r_1}{r_2} = \frac{\vartheta_2}{\vartheta_1} = \frac{\tau_1}{\tau_2} = \frac{\omega_2}{\omega_1} = \frac{\alpha_2}{\alpha_1}$$

# of teeth

Inverse relationship between speed and torque

input

$$\tau_2 = \tau_1 \frac{N_2}{N_1} \qquad TR = \frac{N_1}{N_2}$$

output

mechanical parameter

Equivalent J

$$\ddot{\vartheta}_1 J_1 \Leftarrow \tau_1 = \tau_2 \frac{N_1}{N_2} = \ddot{\vartheta}_2 J_2 \frac{N_1}{N_2}$$

$$J_1 = \frac{\ddot{\vartheta}_2}{\ddot{\vartheta}_1} J_2 \frac{N_1}{N_2} \Rightarrow \left(\frac{N_1}{N_2} \right)^2 J_2$$

$$J_1 = TR^2 J_2$$

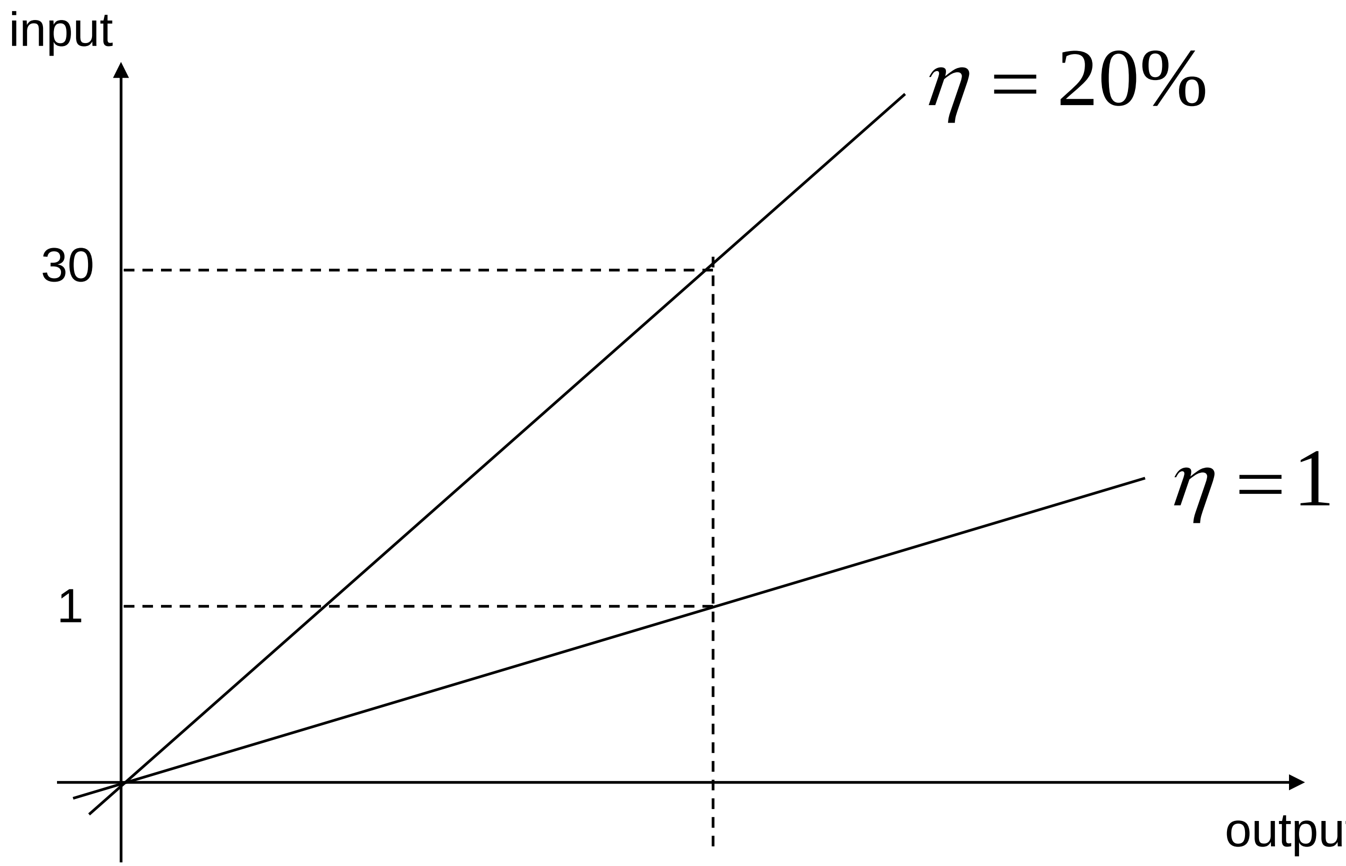
- J as seen from the motor

In reality

$$\tau_2 = \tau_1 \frac{1}{TR} \eta$$

- Where η is the efficiency of the mechanism (from 0 to 1)
- η is related to power, speed ratio doesn't change
- η is also the ratio of input power vs. power at the output

For example



Example

Specifications

reduction ratio (nominal)	weight without motor	length without motor L2	length with motor			output torque continuous operation	output torque intermittent operation	direction of rotation (reversible)	efficiency
			1319 T L1	1331 T L1	1336 U L1	M max.	M max.		
	g	mm	mm	mm	mm	mNm	mNm		%
3,71:1	17	20,9	34,1	45,9	50,9	200	300	=	90
14 :1	20	25,0	38,2	50,0	55,0	300	450	=	80
43 :1	24	29,2	42,4	54,2	59,2	300	450	=	70
66 :1	24	29,2	42,4	54,2	59,2	300	450	=	70
134 :1	27	33,3	46,5	58,3	63,3	300	450	=	60
159 :1	27	33,3	46,5	58,3	63,3	300	450	=	60
246 :1	27	33,3	46,5	58,3	63,3	300	450	=	60
415 :1	30	37,4	50,6	62,4	67,4	300	450	=	55
592 :1	30	37,4	50,6	62,4	67,4	300	450	=	55
989 :1	30	37,4	50,6	62,4	67,4	300	450	=	55
1 526 :1	30	37,4	50,6	62,4	67,4	300	450	=	55

Motion conversion

- Start with

$$\tau_2 = \frac{N_2}{N_1} \tau_1$$

- Design *TR*, more torque (usually)

$$TR < 1$$

$$N_2 > N_1$$

$$J_1 < J_2 \Leftrightarrow \omega_2 < \omega_1$$

Viscous friction

- Easy:

$$\tau_{viscous} = B_2 \dot{\vartheta}_2$$

$$\tau_{eq\_viscous} = TR \cdot \tau_{viscous} = TR \cdot B_2 \dot{\vartheta}_2$$

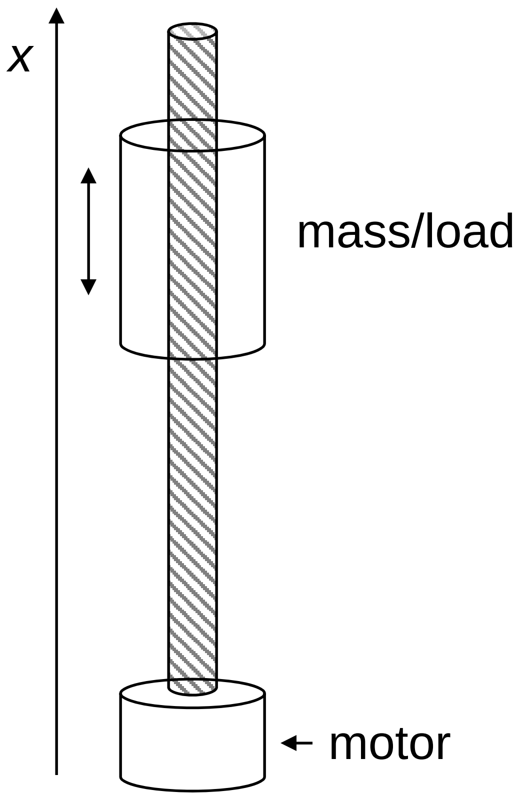
$$B_{eq} \dot{\vartheta}_1 = TR \cdot B_2 \dot{\vartheta}_2 \Rightarrow B_{eq} = TR^2 B_2$$

- Coulomb friction:

$$\tau_{eq} = TR \cdot F_c \operatorname{sgn}(\dot{\vartheta}_2)$$

Lead screw

- Rotary to linear motion conversion (P=pitch in #of turns/mm or inches)



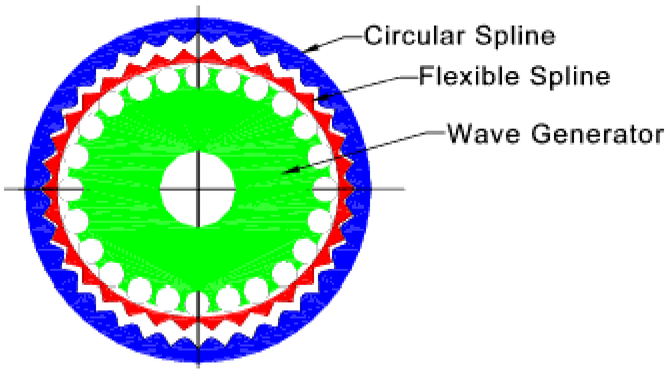
$$\vartheta[rad] = 2\pi P x$$

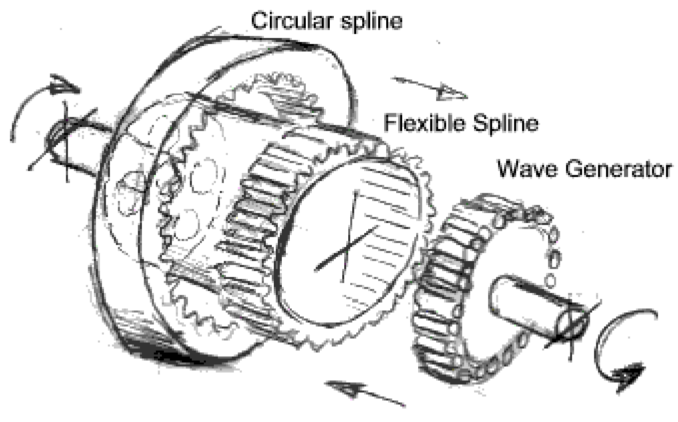
$$\dot{\vartheta} = 2\pi P \dot{x}$$

$$E_{rot} = E_{lin} \Rightarrow \frac{1}{2} M_{load} v^2 = \frac{1}{2} J \omega^2 \Rightarrow$$

$$\Rightarrow J = \frac{M_{load}}{(2\pi P)^2}$$

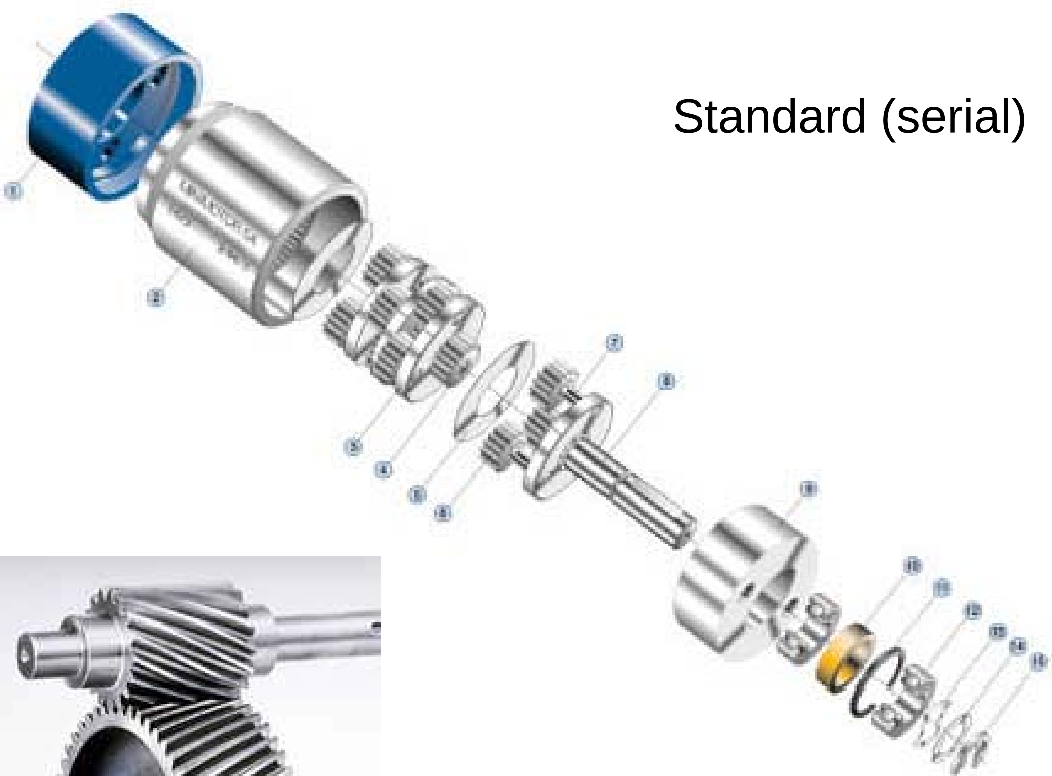
Harmonic drives



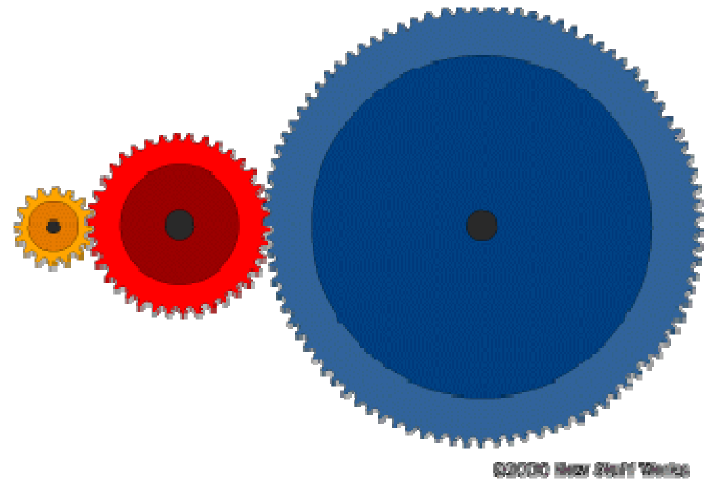


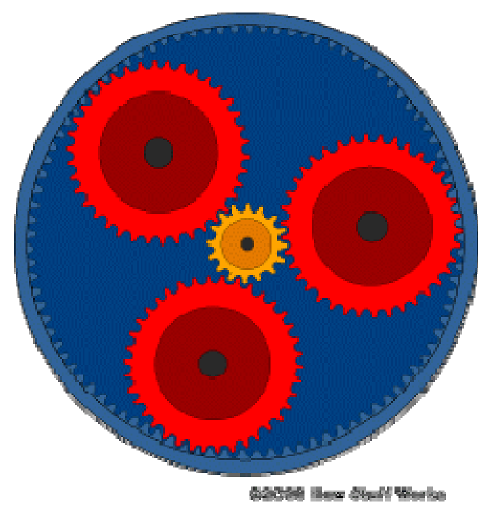
From the harmonic drive website
http://www.harmonicdrive.de

Gearhead (for real)



Standard (serial)





Planetary

Example

- Designing the single joint
  - Given:

$$\ddot{\vartheta}_{\max} \Rightarrow \tau = J_{eq}\ddot{\vartheta} \Rightarrow \tau_{\max} = J_{eq}\ddot{\vartheta}_{\max} = J_{load}TR^2\ddot{\vartheta}_{\max}$$

- Then taking into account some more realistic components:

$$\tau_{\max} = J_{load}\frac{TR^2}{\eta}\ddot{\vartheta}_{\max}$$

Example (continued)

$$\tau_{\max} = J_{load} \frac{TR^2}{\eta} \ddot{\vartheta}_{\max}$$

$$P = \tau_{\max} \dot{\vartheta} \Rightarrow \text{given } \dot{\vartheta}_{\max} \Rightarrow \text{get } P$$

motor power, from catalog

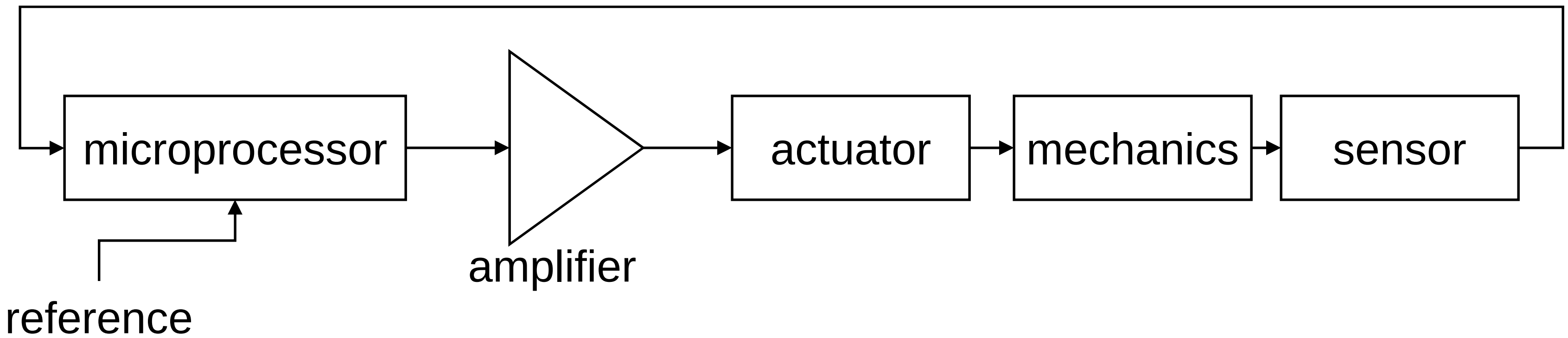
This guarantees that the motor can still deliver maximum torque at maximum speed

More on real world components

- Efficiency
- Eccentricity
- Backlash
- Vibrations

- To get better results during design mechanical systems can be simulated

Control of a single joint



Components

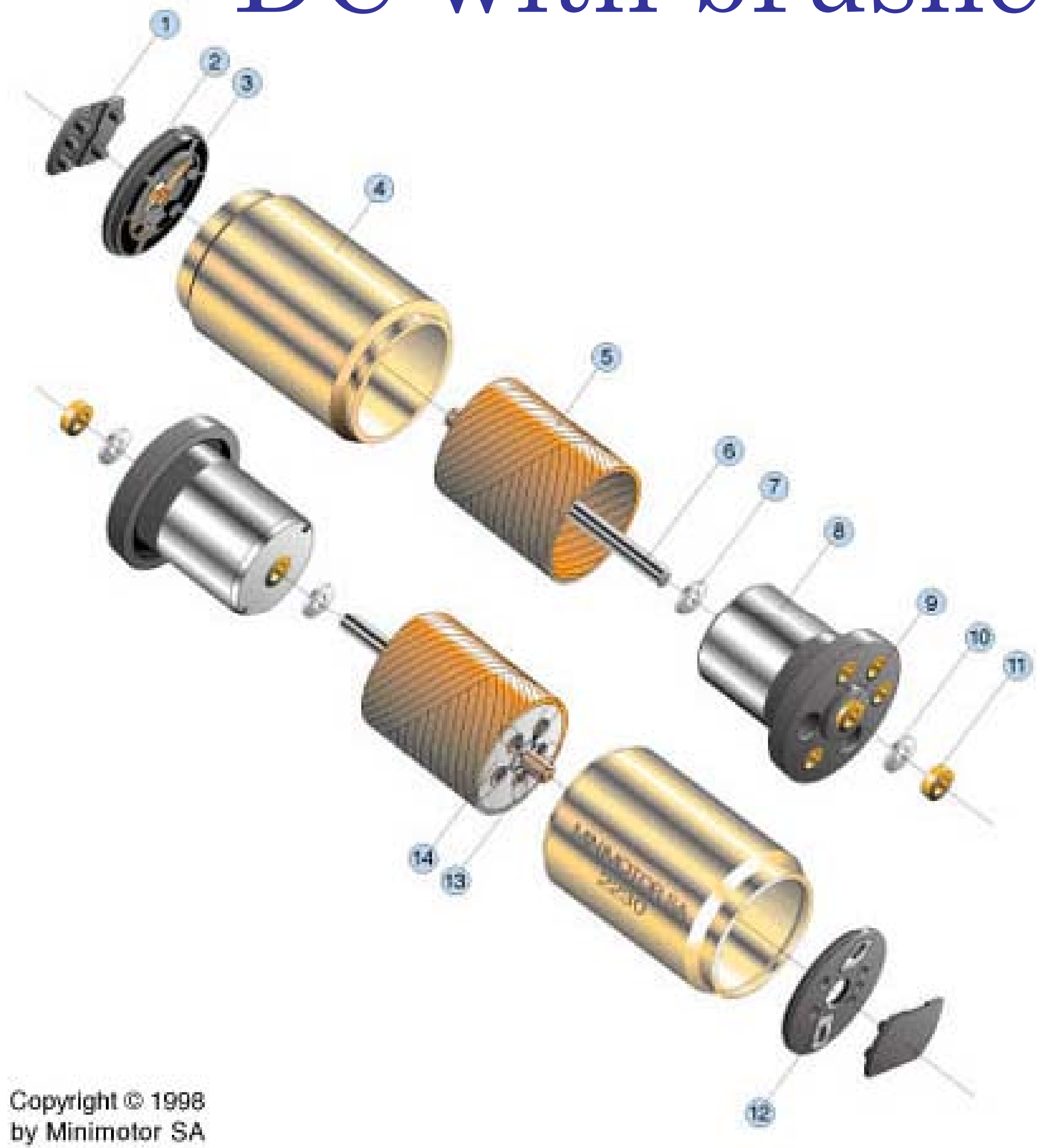
- Digital microprocessor:
  - Microcontroller, processor + special interfaces
- Amplifier (drives the motor)
  - Turns control signals into power signals
- Actuator
  - E.g. electric motor
- Mechanics/load
  - The robot!
- Sensors
  - For intelligence

Actuators

- Various types:
  - AC, DC, stepper, etc.
  - DC
    - Brushless
    - With brushes
- We'll have a look at the DC with brushes, simple to control, widely used in robotics

DC-brushless

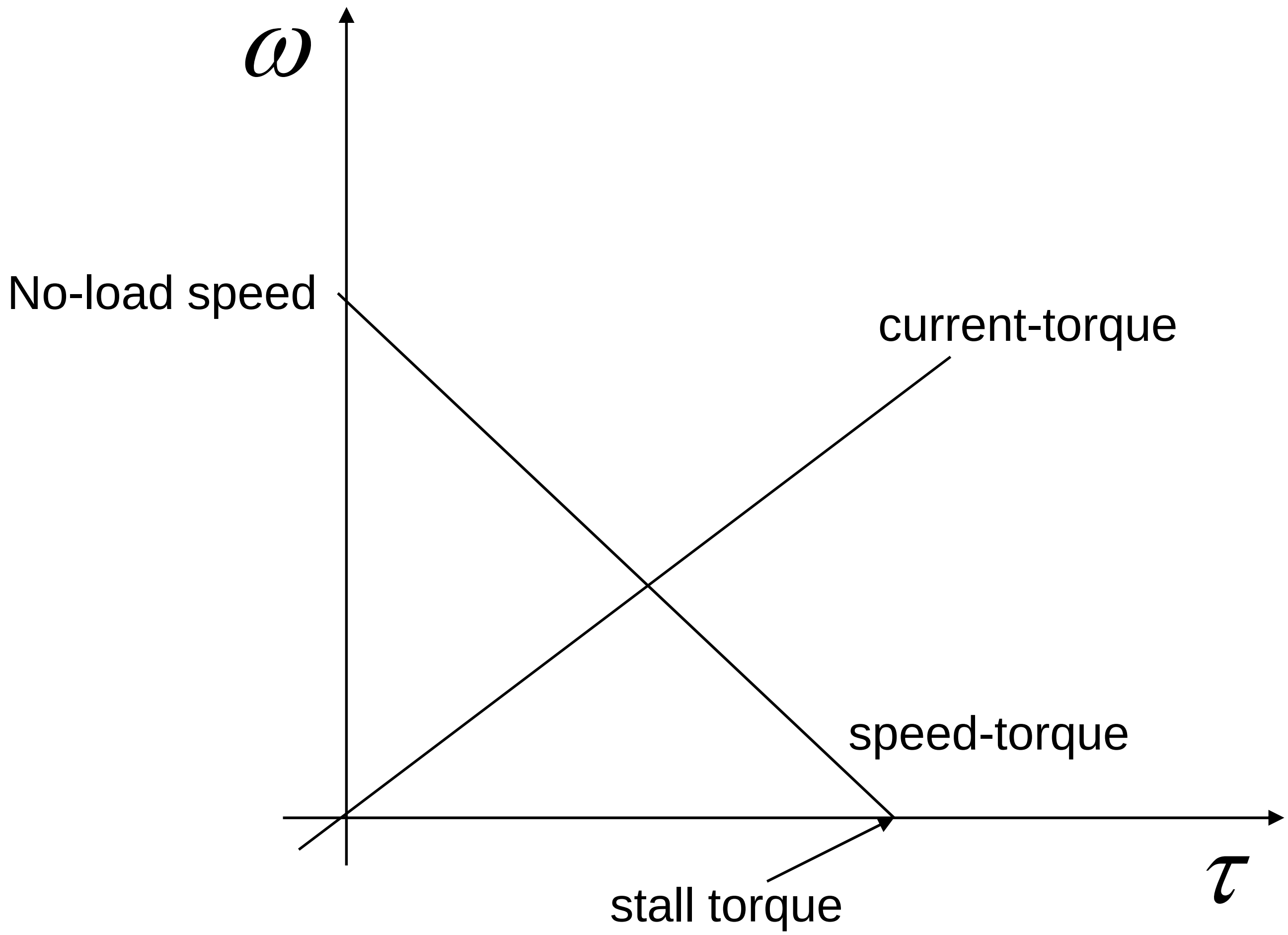
DC with brushes



Modeling the DC motor

- Speed-torque and torque-current relationships are linear

In particular



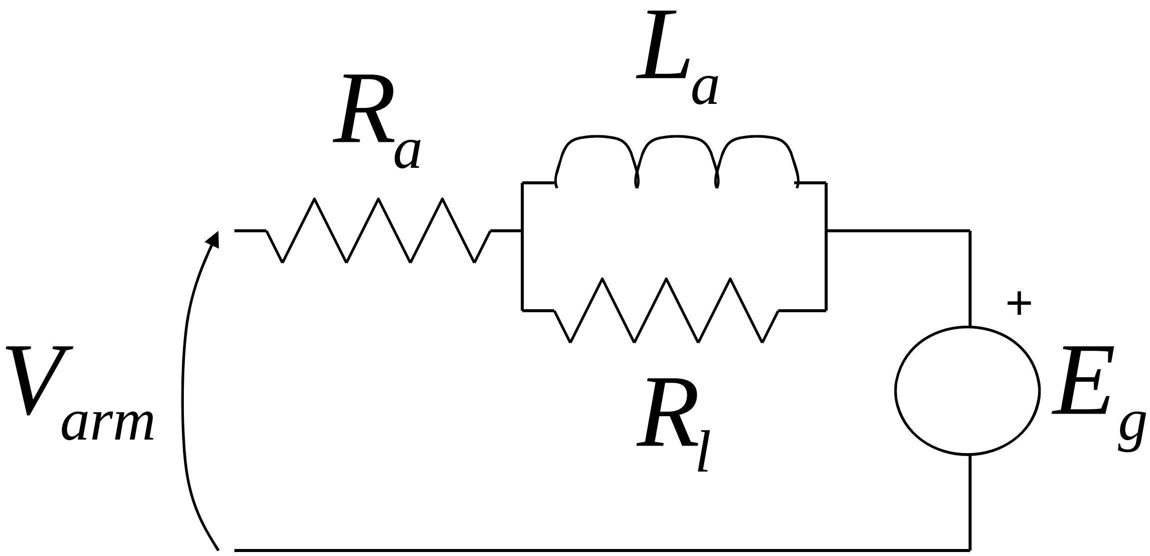
Real numbers!

http://www.minimotor.ch

Series 1331 … SR

		1331 T		006 SR	012 SR	024 SR	
1	Nominal voltage	U_N		6	12	24	Volt
2	Terminal resistance	R		2,83	13,7	52,9	Ω
3	Output power	$P_{2\,max.}$		3,11	2,57	2,66	W
4	Efficiency	$\eta_{max.}$		81	80	80	%
5	No-load speed	n_o		10 600	9 900	10 400	rpm
6	No-load current (with shaft ø 1,5 mm)	I_o		0,0220	0,0105	0,0055	A
7	Stall torque	M_H		11,20	9,90	9,76	mNm
8	Friction torque	M_R		0,12	0,12	0,12	mNm
9	Speed constant	k_n		1 790	835	439	rpm/V
10	Back-EMF constant	k_E		0,56	1,20	2,28	mV/rpm
11	Torque constant	k_M		5,35	11,4	21,8	mNm/A
12	Current constant	k_I		0,187	0,087	0,046	A/mNm
13	Slope of n-M curve	$\Delta n/\Delta M$		946	1 000	1 070	rpm/mNm
14	Rotor inductance	L		70	310	1 100	µH
15	Mechanical time constant	τ_m		7	7	7	ms
16	Rotor inertia	J		0,71	0,67	0,63	gcm^2
17	Angular acceleration	$\alpha_{max.}$		160	150	160	$\cdot 10^3 rad/s^2$
18	Thermal resistance	$R_{th\,1} / R_{th\,2}$	6 / 25				K/W
19	Thermal time constant	τ_{w1} / τ_{w2}	5 / 190				s
20	Operating temperature range:						
	– motor		– 30 … + 85 (optional – 55 … + 125)				°C
	– rotor, max. permissible		+ 125				°C

Electrical diagram



$$E_g = \omega(t)K_E$$

Meaning of components

R_a	• Armature resistance (including brushes)
V_{arm}	• Armature voltage
R_l	• Losses due to magnetic field
E_g	• Back EMF produced by the rotation of the armature in the field
L_a	• Coil inductance

We can write…

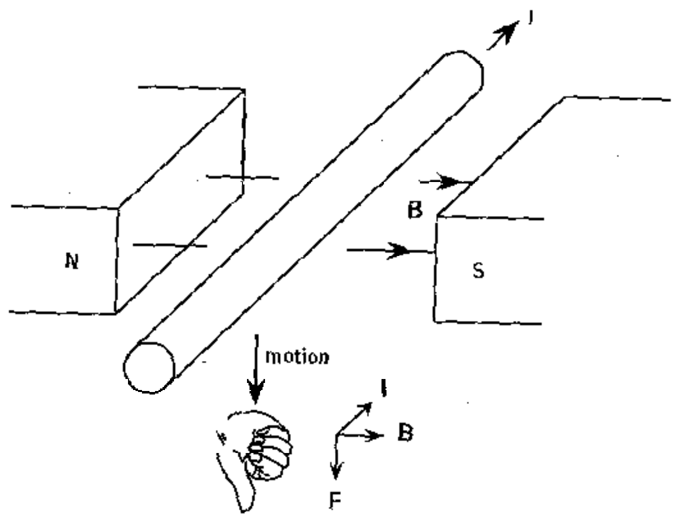
$$V_{arm} = R_a I_a + L_a \dot{I}_a + \omega(t) K_E$$

$$\text{for } R_l \ll R_a$$

which is the case at the frequency of interest, and we also have…

$$\tau = K_T I_a$$

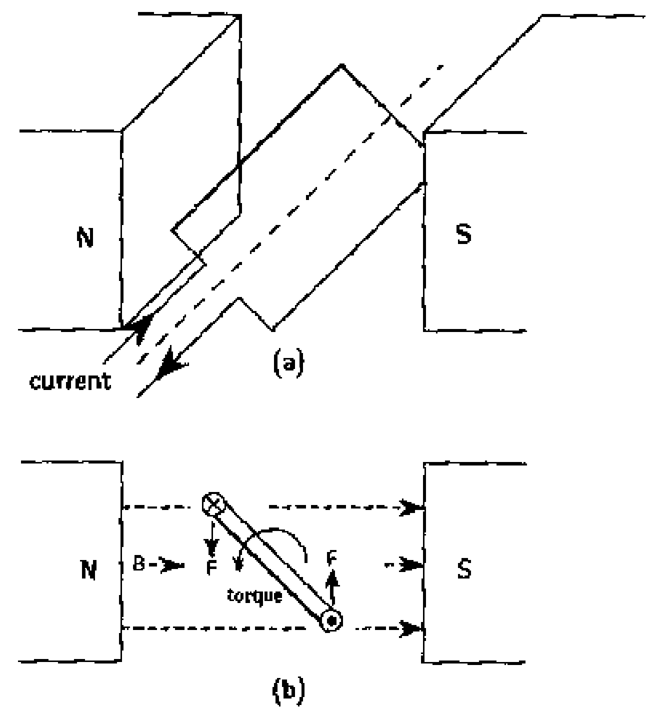
On torque and current



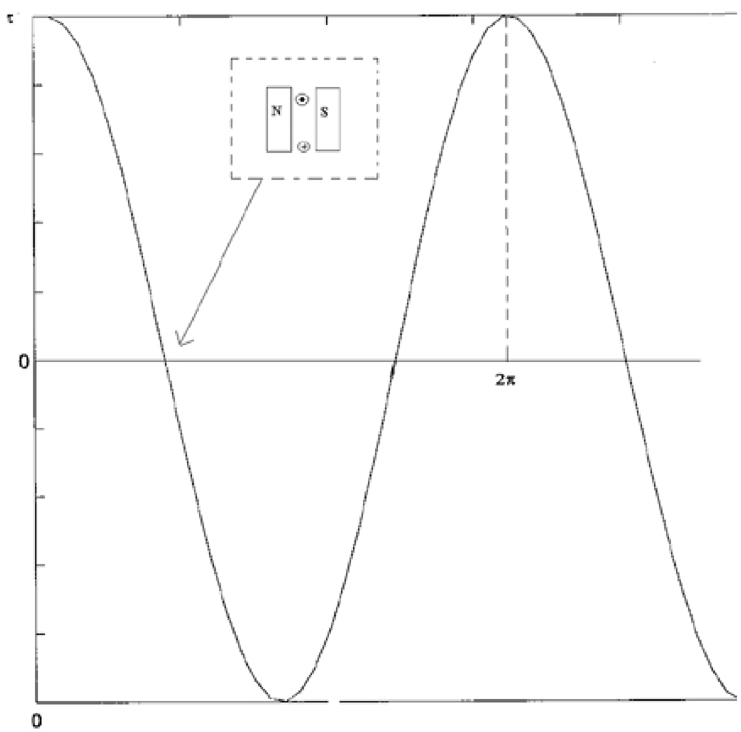
$$F = i\vec{L} \times \vec{B}$$

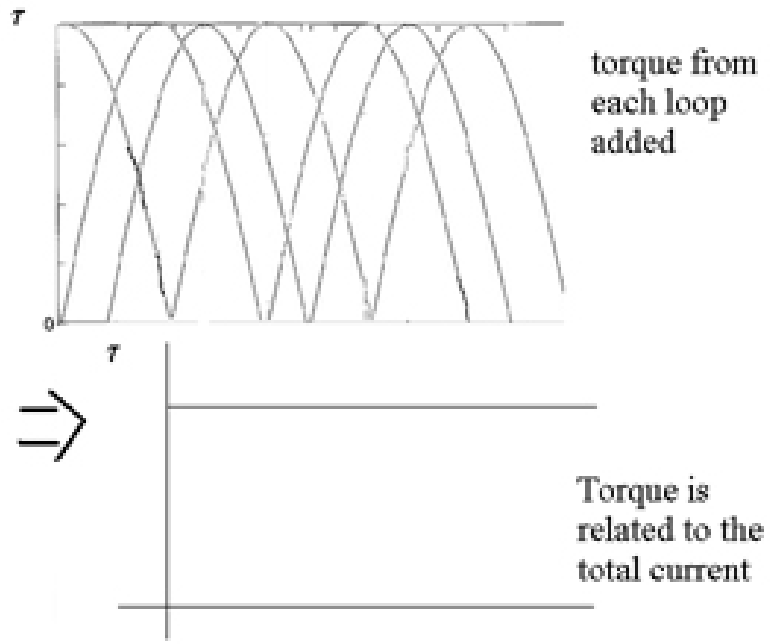
↑

$$F_{lorentz} = q\vec{v} \times \vec{B}$$



Thus for many coils…





Back to motor modeling…

$$\tau = (J_M + J_L)\dot{\omega}(t) + B\omega(t) + \tau_f + \tau_{gr}$$

τ	• Torque generated
J_M	• Inertia of the motor
J_L	• Inertia of the load
τ_f	• Friction
τ_{gr}	• Gravity

Furthermore…

$$V_{arm} = R_a I_a + L_a \dot{I}_a + \omega(t) K_E$$

$$\tau = K_T I_a$$

$$\tau = (J_M + J_L)\dot{\omega}(t) + B\omega(t) + \tau_f + \tau_{gr}$$

Consequently

$$\begin{bmatrix} \dot{I}_a \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} R_a / L_a & K_E / L_a \\ K_T / J_M + J_L & B / J_M + J_L \end{bmatrix} \cdot \begin{bmatrix} I_a \\ \omega \end{bmatrix} + \begin{bmatrix} -V_{arm} / L_a \\ \tau_f + \tau_{gr} / J_M + J_L \end{bmatrix}$$

- A linear system of two equations (differential)
- Q: can you write a transfer function from these equations?
- Q: can you transform the equations into a block diagram?

By Laplace-transforming

$$V_{arm}(s) = R_a I_a(s) + L_a I_a(s)s + \omega(s)K_E \Rightarrow I_a(s) = \frac{V_{arm}(s) - \omega(s)K_E}{R_a + L_a s}$$

$$\tau = K_T I_a$$

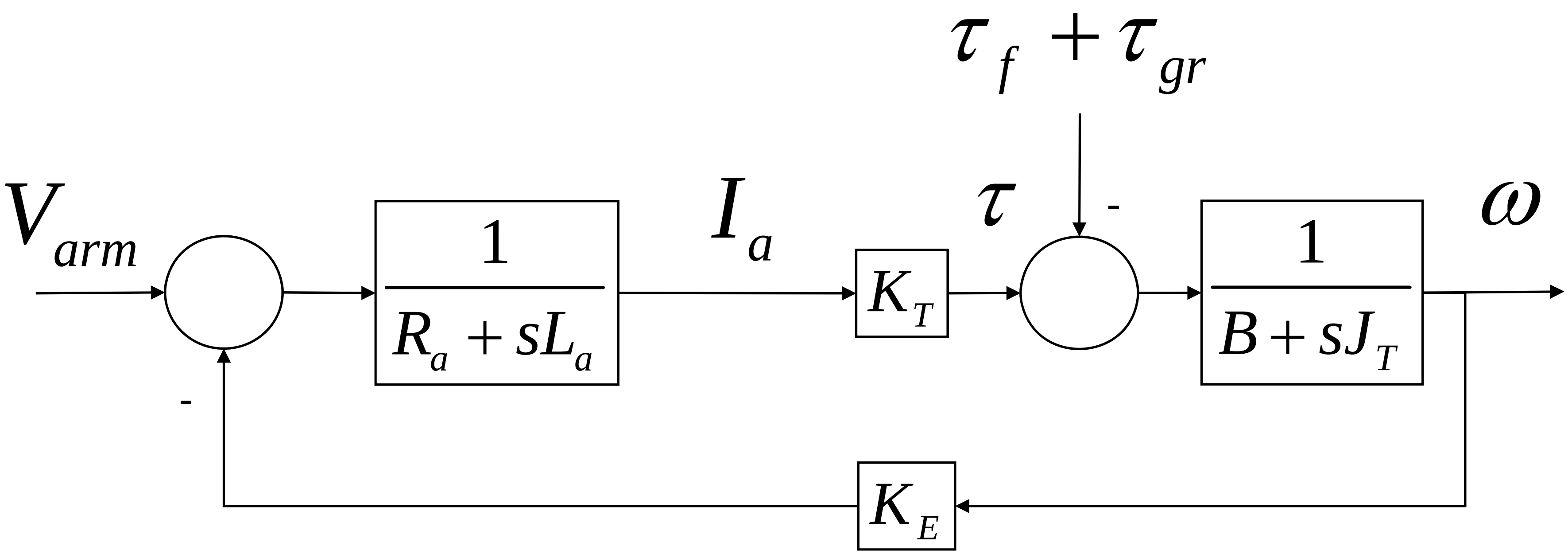
$$K_T \frac{V_{arm}(s) - \omega(s)K_E}{R_a + L_a s} = (J_M + J_L)\omega(s)s + B\omega(s) + \tau_f + \tau_{gr}$$

and finally

$$\frac{\omega(s)}{V_{arm}(s)} = \frac{K_T / L_a J_T}{s^2 + [(R_a J_T + L_a B)/L_a J_T]s + (K_T K_E + R_a B)/L_a J_T}$$

- Considering gravity and friction as additional inputs

Block diagram

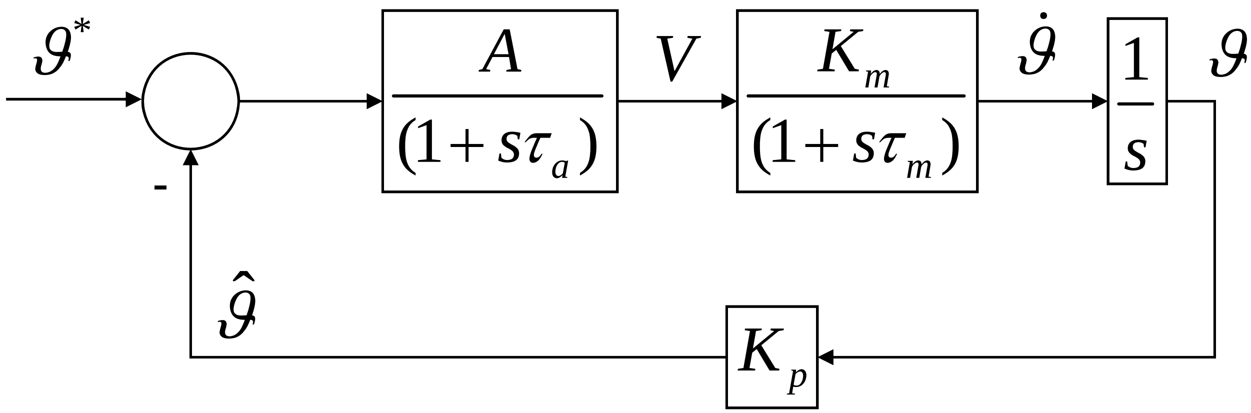


Analysis tools

- Control: determine V_a so to move the motor as desired

- Root locus
- Frequency response

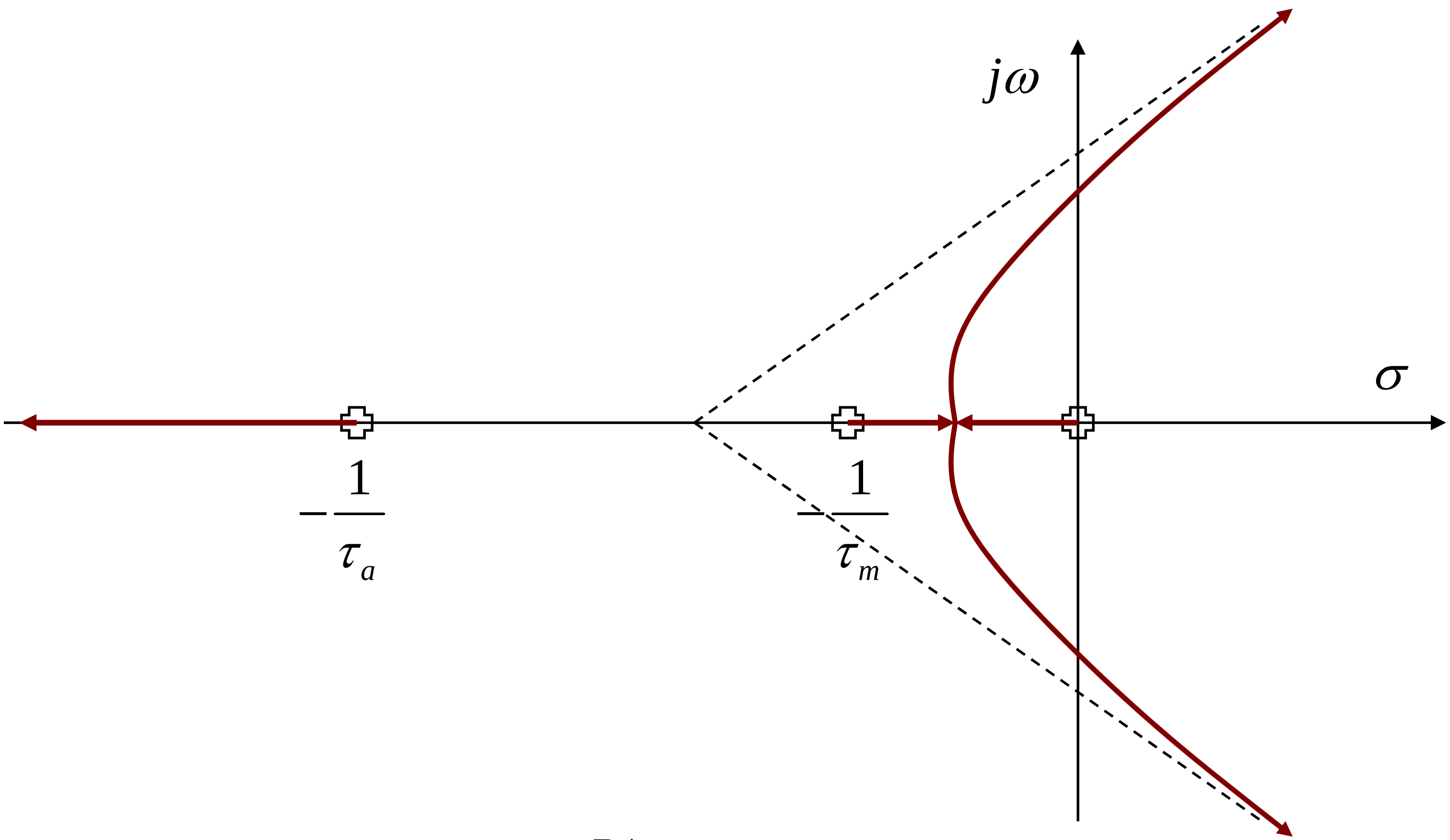
First block diagram



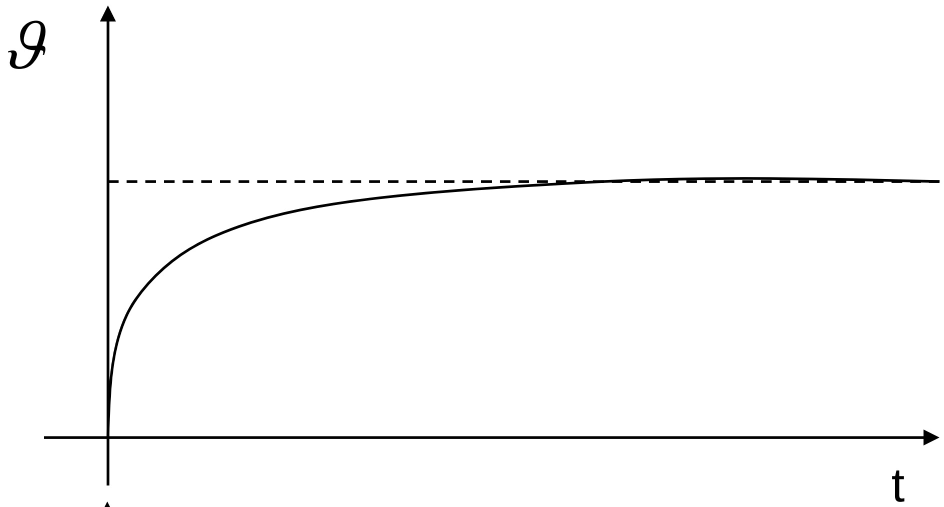
$$H_{open\_loop} = \frac{A}{1+s\tau_a}\frac{K_m}{1+s\tau_m}\frac{K_p}{s}$$

Root locus

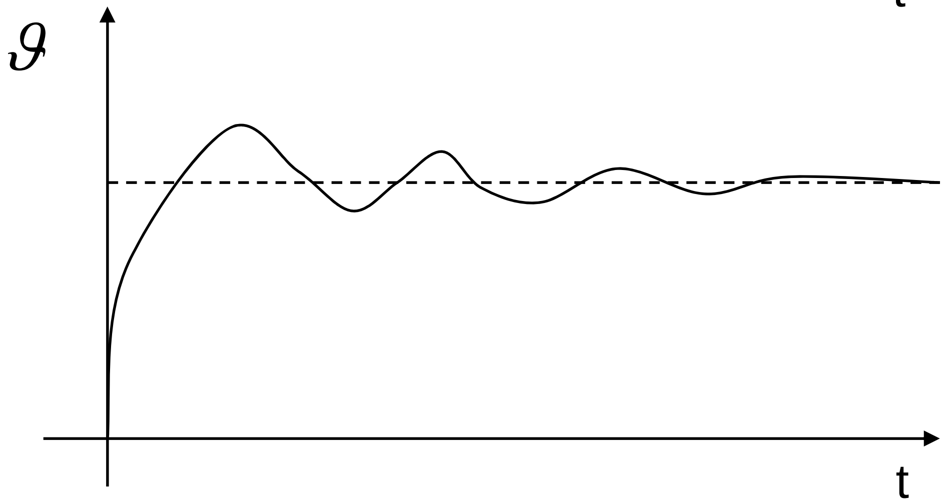
$$H_{open\_loop} = \frac{A}{1+s\tau_a}\frac{K_m}{1+s\tau_m}\frac{K_p}{s} \qquad K = AK_mK_p$$



Changing K

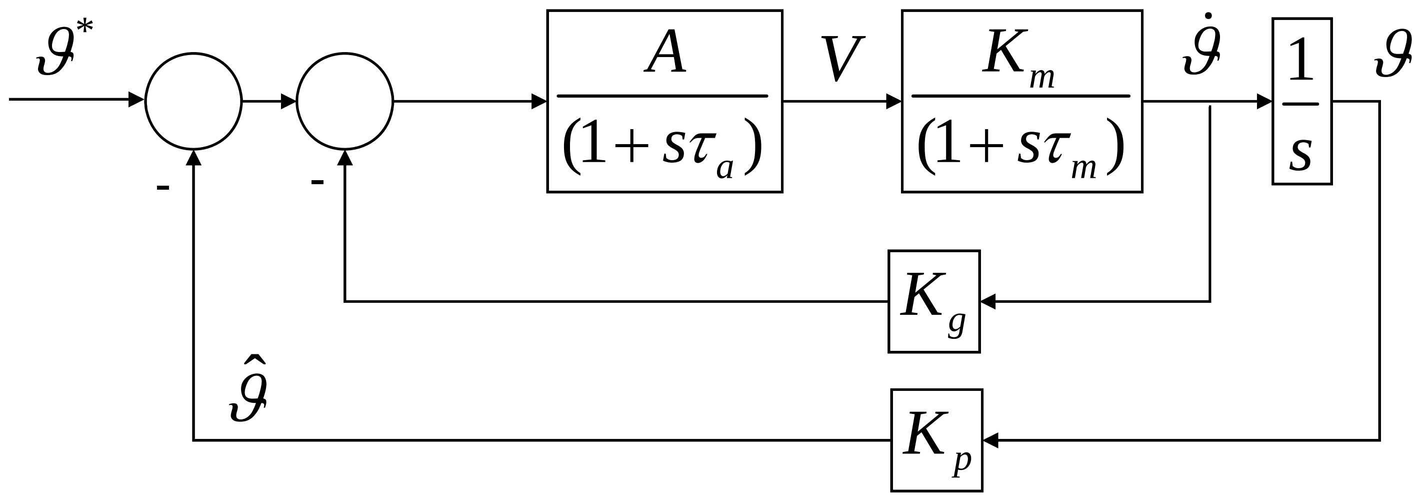


small K



higher K

Let's add something second diagram



$$H_{open\_loop} = \frac{AK_m(K_p + sK_g)}{(1+s\tau_a)(1+s\tau_m)s}$$

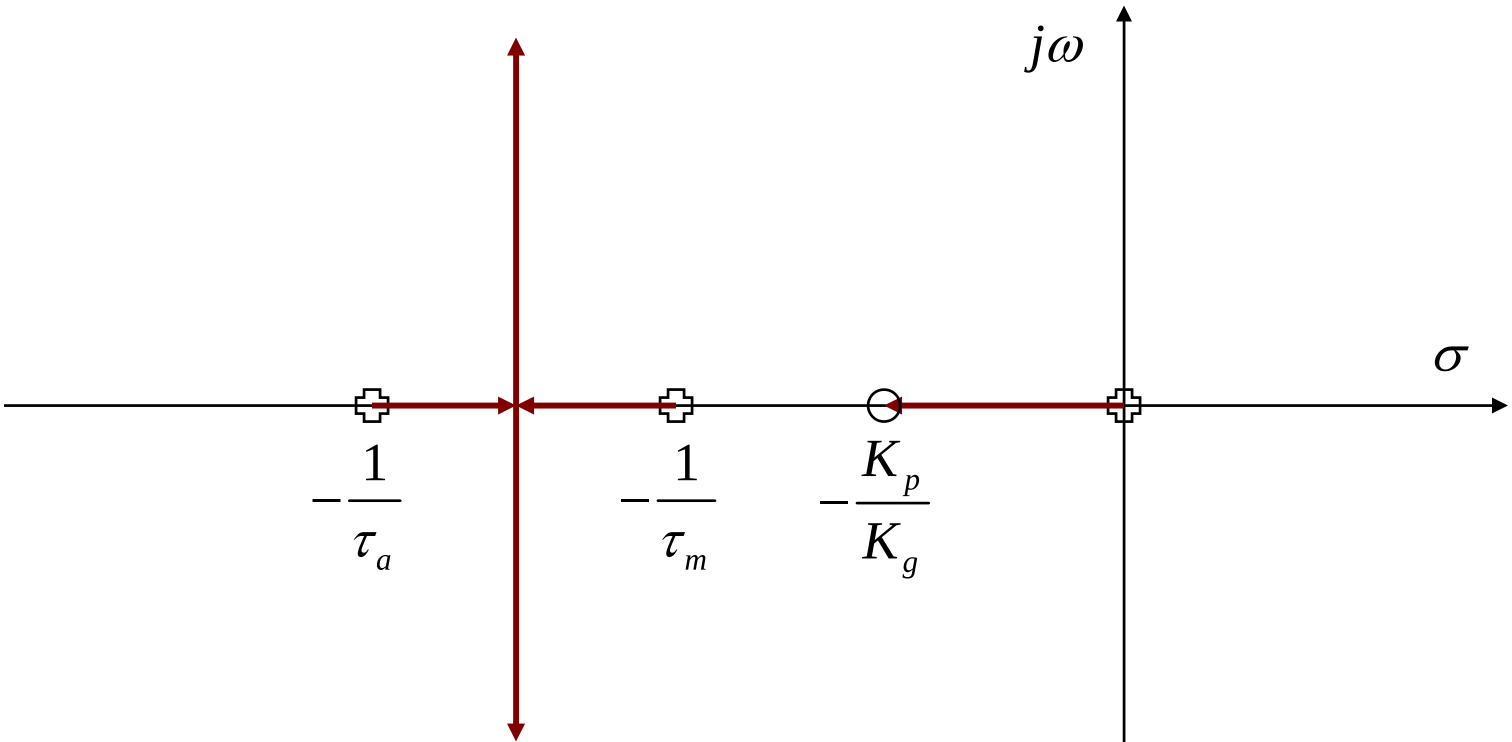
Analysis

$$H_{open\_loop} = \frac{AK_m K_p (1 + s \, {K_g}/{K_p})}{(1 + s\tau_a)(1 + s\tau_m)s}$$

$$K = AK_m K_p$$

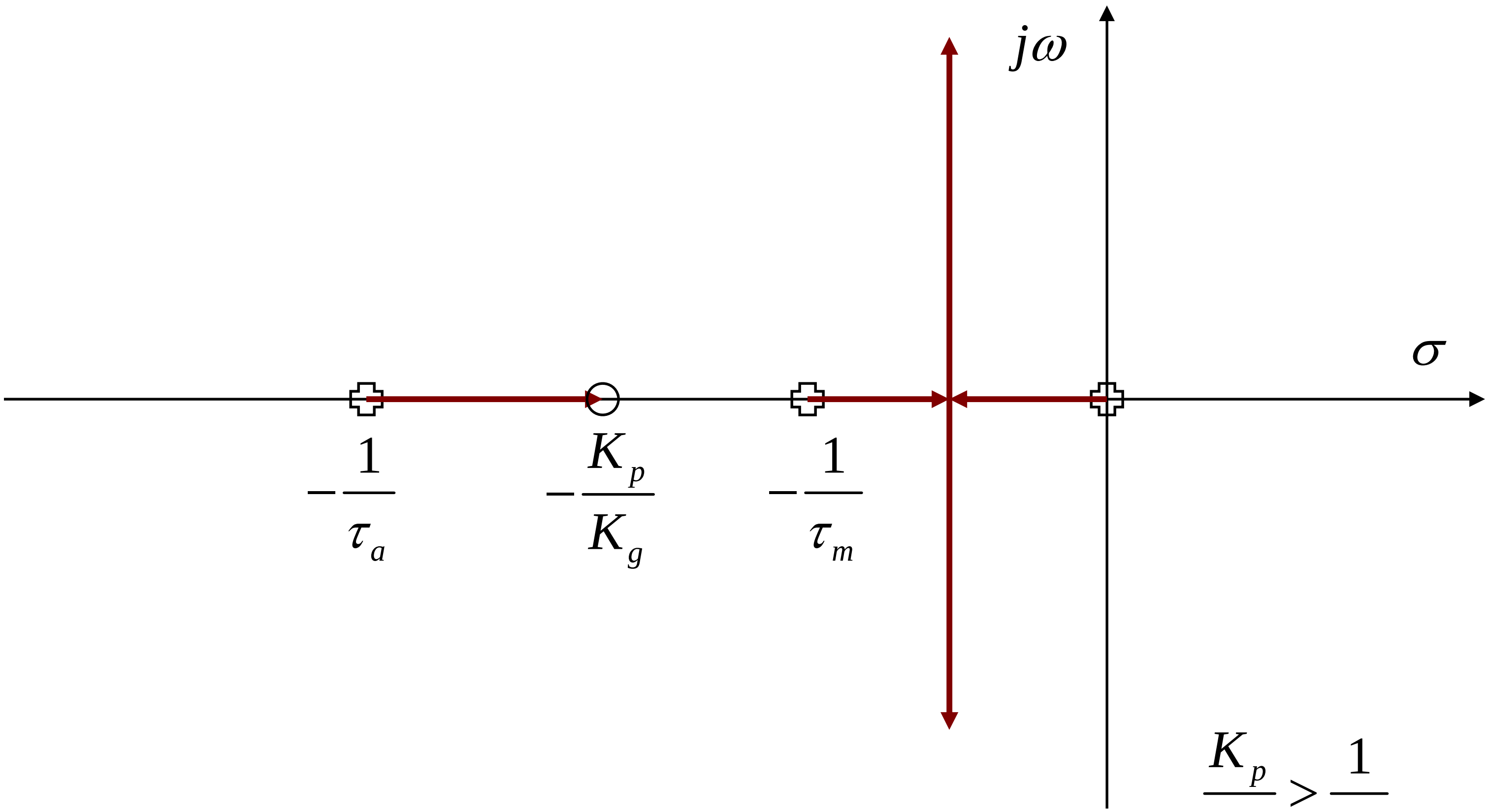
$$Z_{feedback} = \frac{K_g}{K_p}$$

Root locus (case 1)



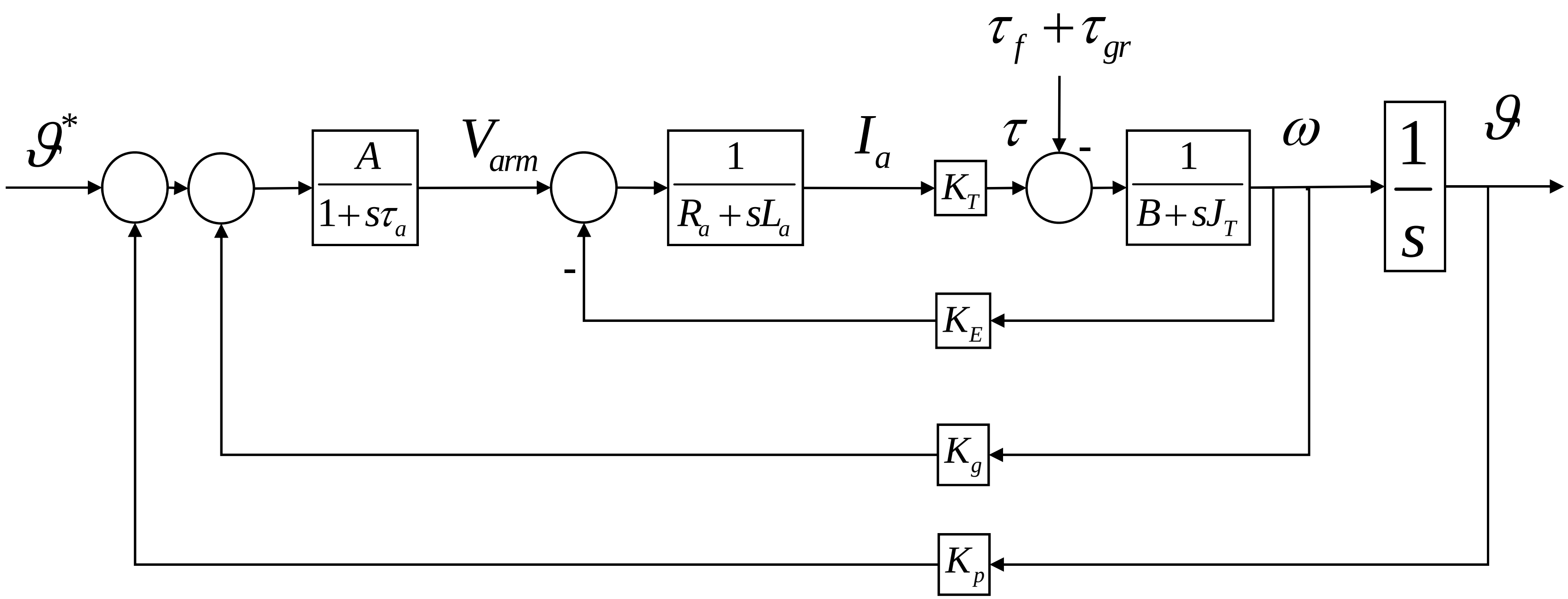
$$\frac{K_p}{K_g} < \frac{1}{\tau_m}$$

Root locus (case 2)



$$\frac{K_p}{K_g} > \frac{1}{\tau_m}$$

Overall...


ϑ^*
A
$1+s\tau_a$
V_{arm}
-
1
R_a+sL_a
I_a
K_T
τ
$\tau_f+\tau_{gr}$
-
1
$B+sJ_T$
ω
1
s
ϑ
K_E
K_g
K_p