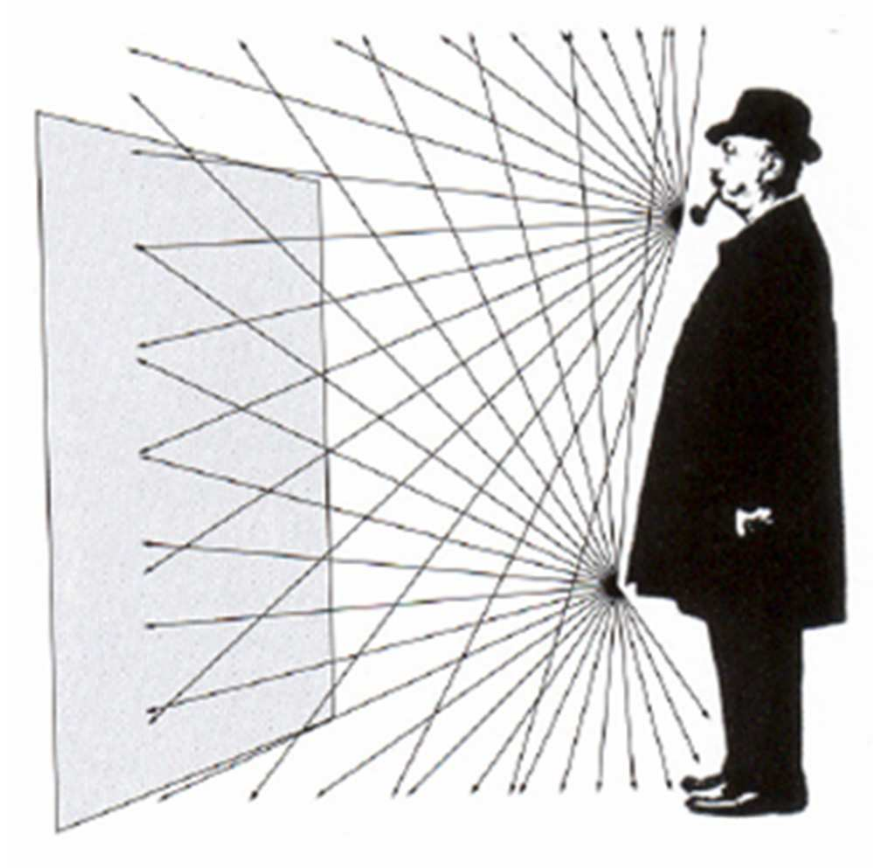
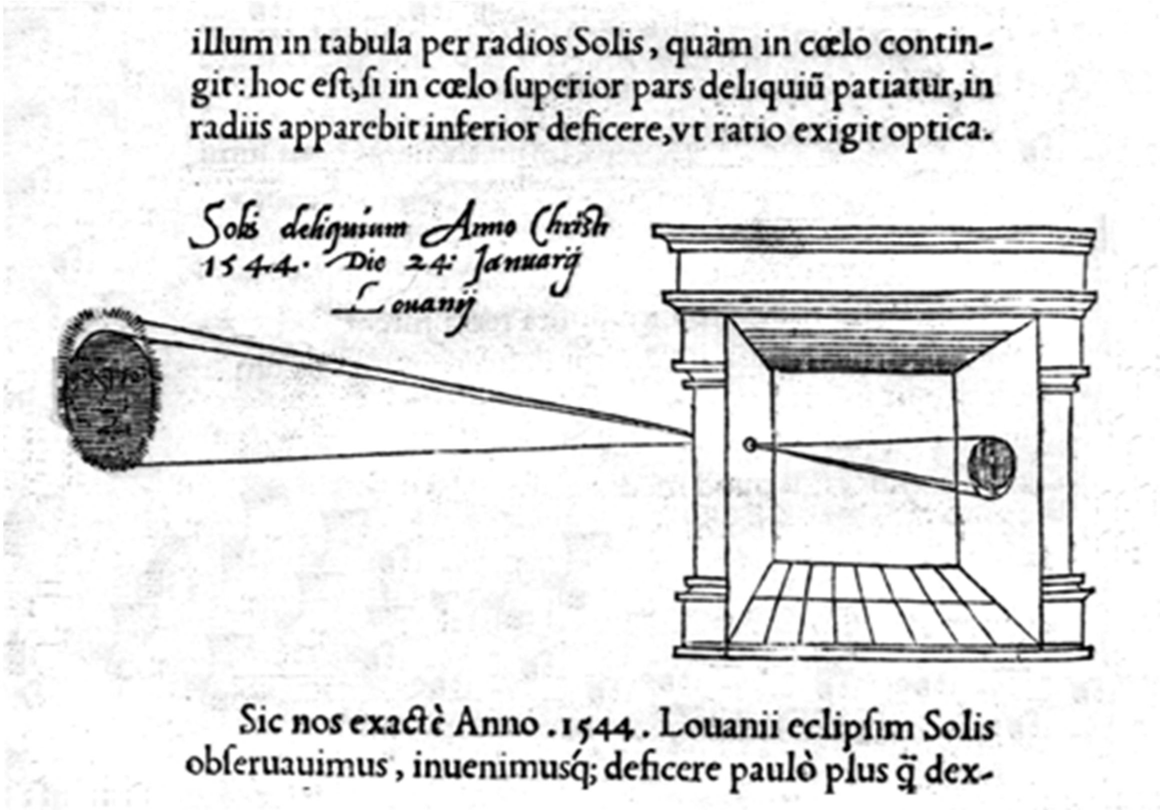
Image formation

Why there is no image on a white paper



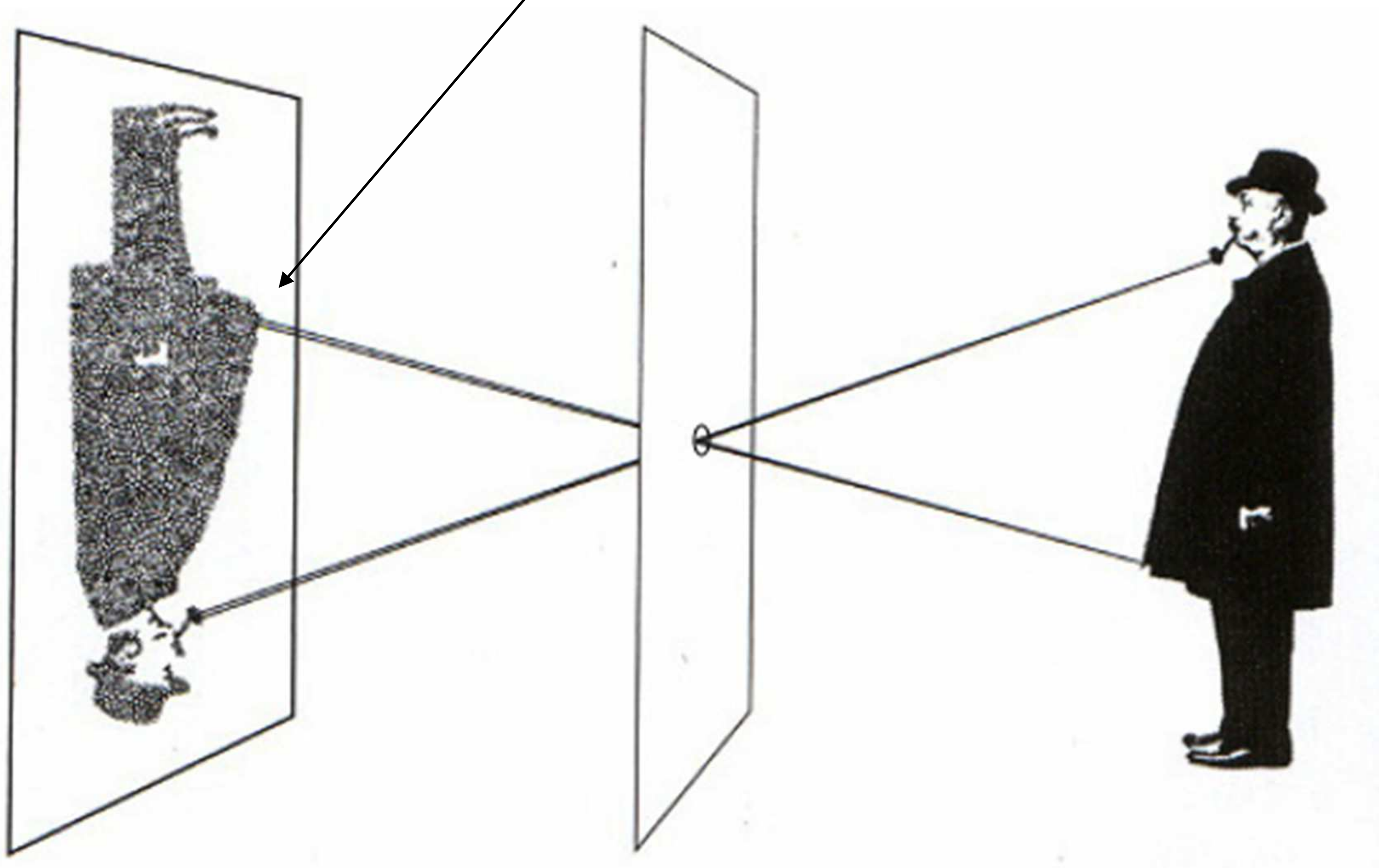
Pinhole



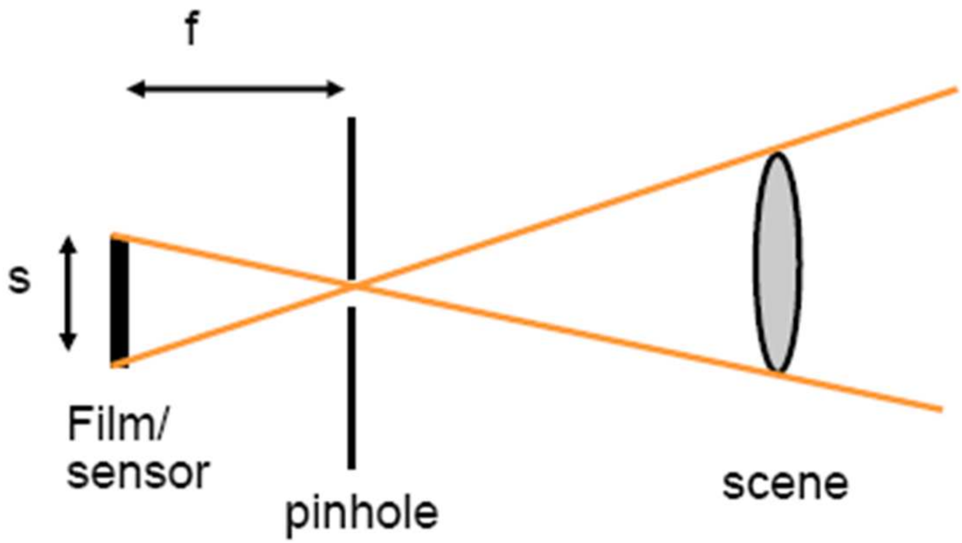
illum in tabula per radios Solis, quàm in cœlo contingit: hoc eſt, ſi in cœlo ſuperior pars deliquiũ patiatur, in radiis apparebit inferior deficere, vt ratio exigit optica.

Sic nos exactè Anno .1544. Louanii eclipſim Solis obſeruauimus, inuenimusq; deficere paulò plus q̃ dex-

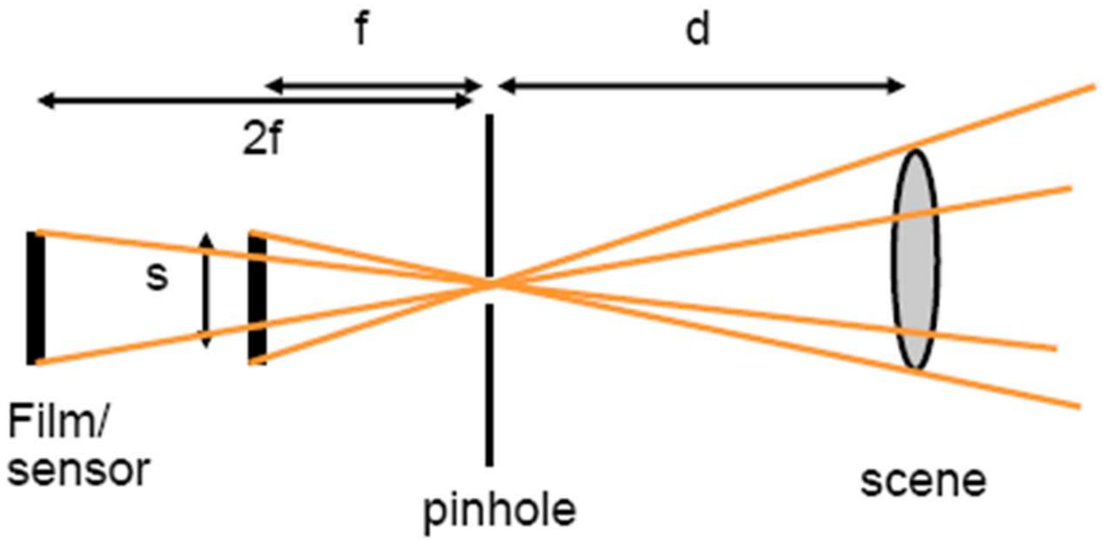
Each point in the scene projects to a single (or very small) point in the image



- The focal length *f* is the distance between the pinhole and the sensor

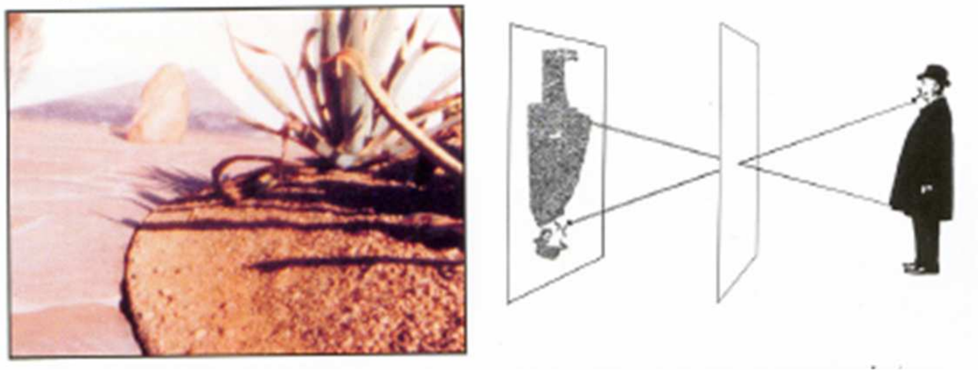


- If we double *f* we double the size of the projected object

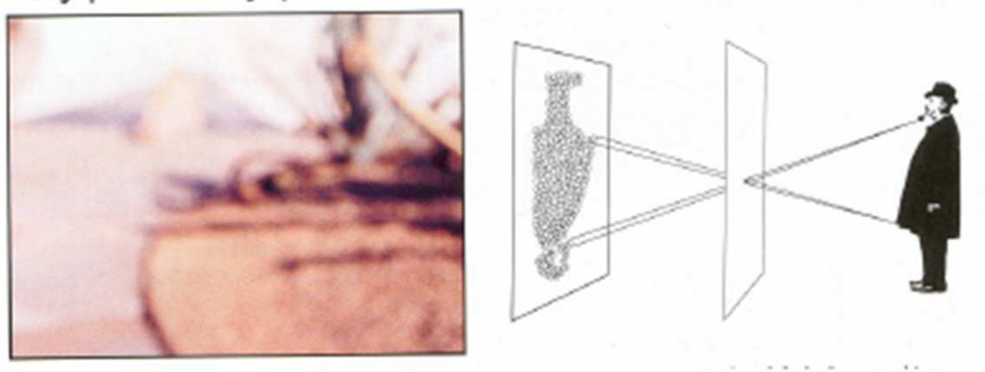


Problems:
- limited light
- the size of the pinhole limits sharpness

Photograph made with small pinhole

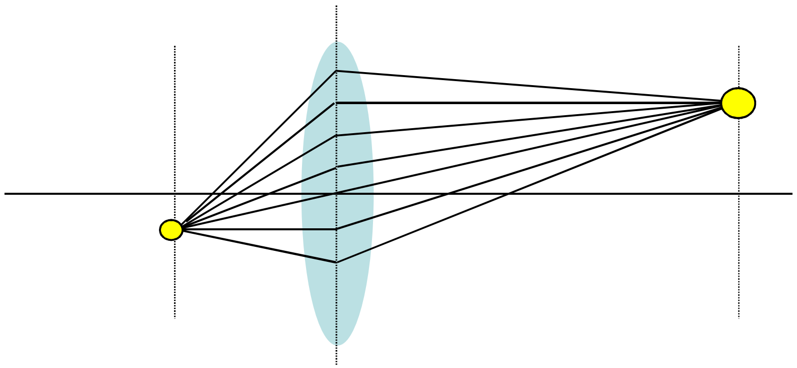


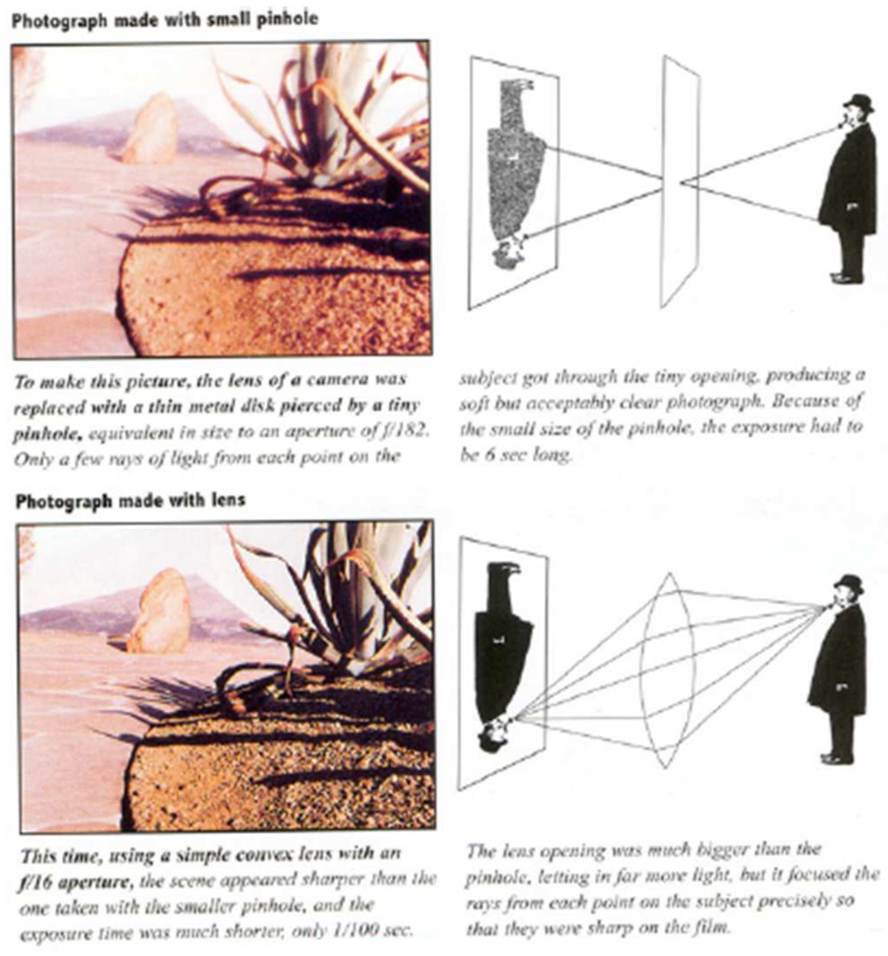
Photograph made with larger pinhole



Converging lenses

Lenses focus the light from different directions/rays (*refraction*)




Photograph made with small pinhole

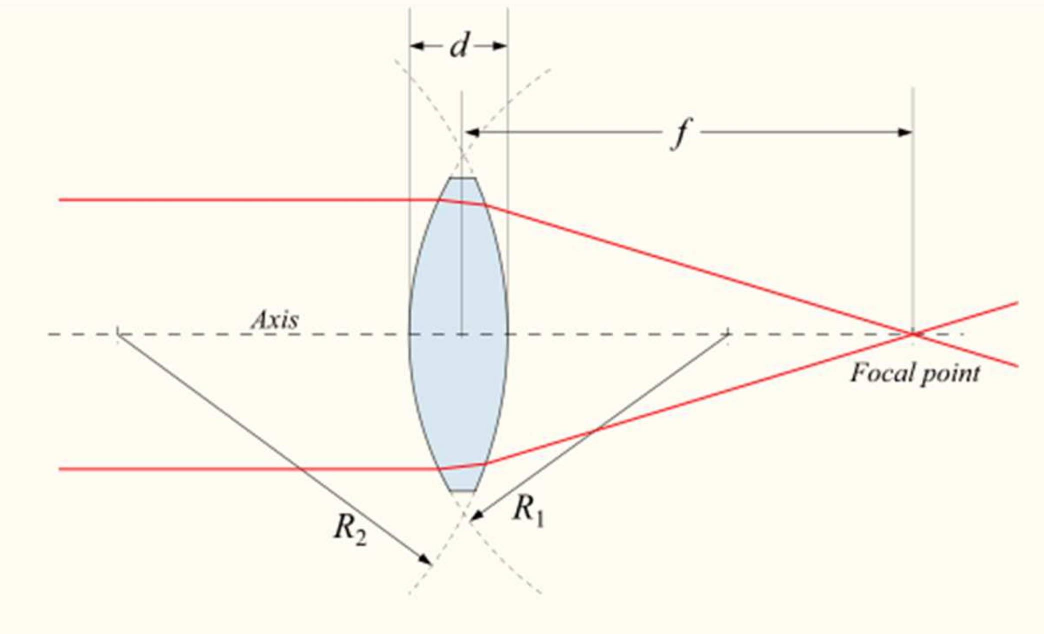
To make this picture, the lens of a camera was replaced with a thin metal disk pierced by a tiny pinhole, *equivalent in size to an aperture of f/182. Only a few rays of light from each point on the subject got through the tiny opening, producing a soft but acceptably clear photograph. Because of the small size of the pinhole, the exposure had to be 6 sec long.*

Photograph made with lens

This time, using a simple convex lens with an f/16 aperture, *the scene appeared sharper than the one taken with the smaller pinhole, and the exposure time was much shorter, only 1/100 sec. The lens opening was much bigger than the pinhole, letting in far more light, but it focused the rays from each point on the subject precisely so that they were sharp on the film.*

How to draw the rays

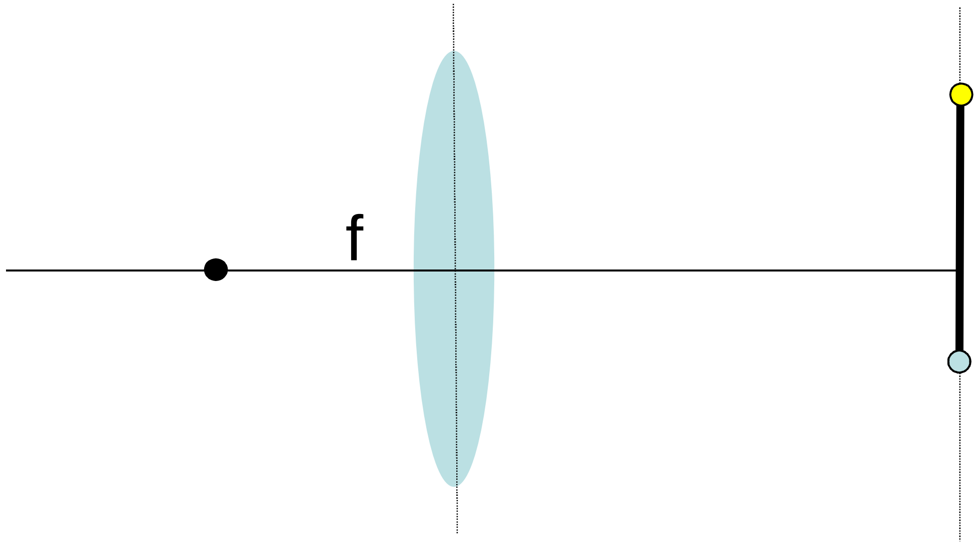
- Three rules
  1. incident rays parallel to the principal axis converge to the focal point
  2. incident rays passing through the center of the lens do not modify their direction
  3. incident rays through the focal point on the right side of the lens get reflected and travel parallel to the principal axis



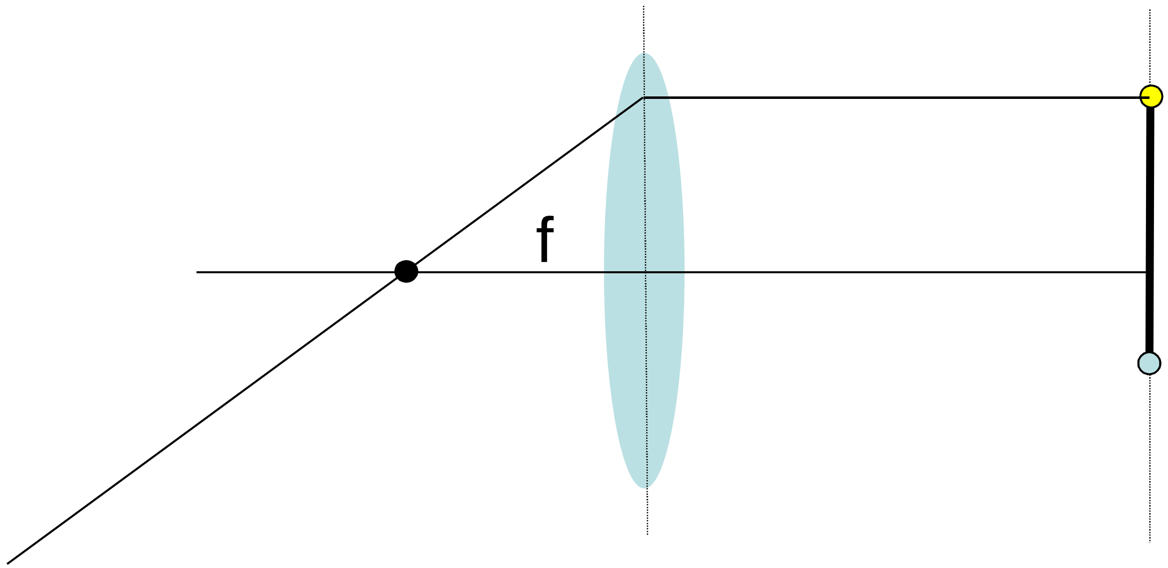
Positive (converging) lens

Thin lens approx:
d small compared to R1 and R2

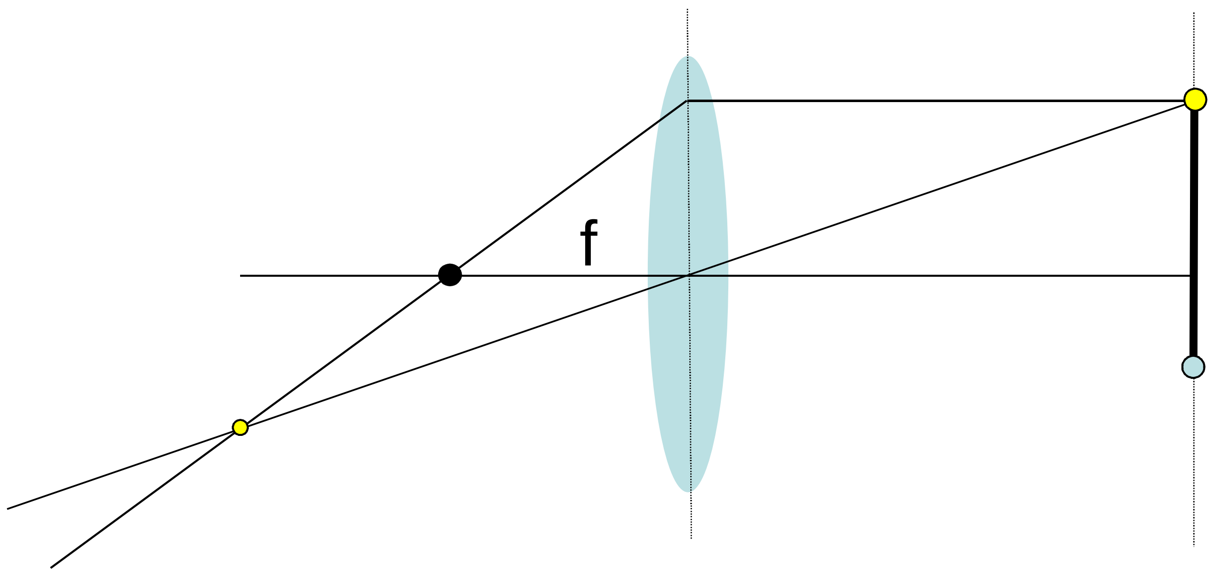
Example



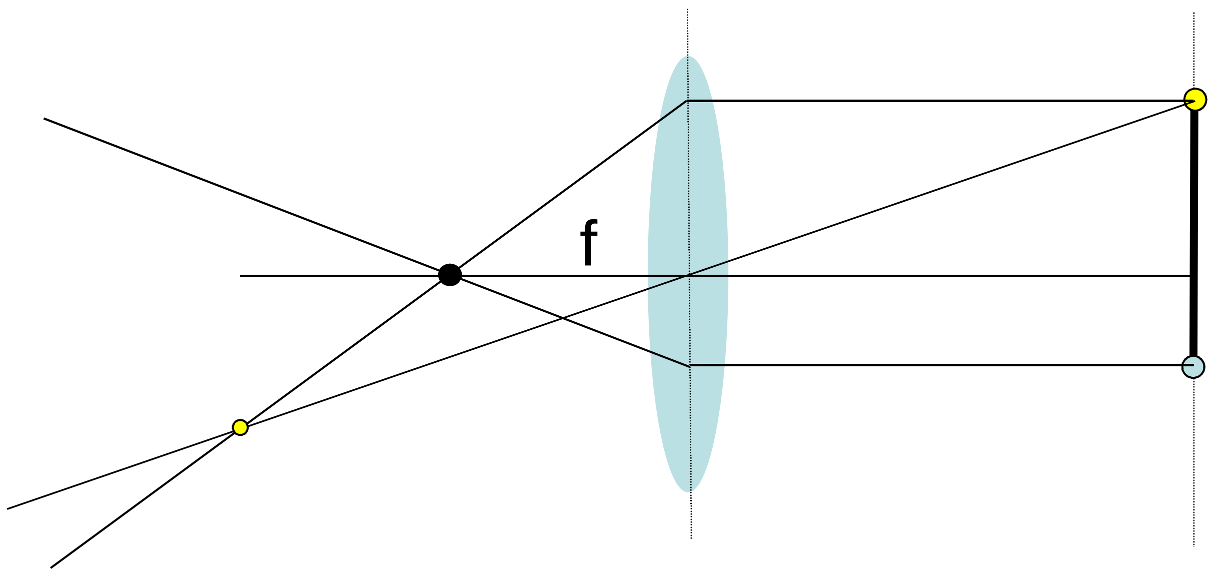
Example



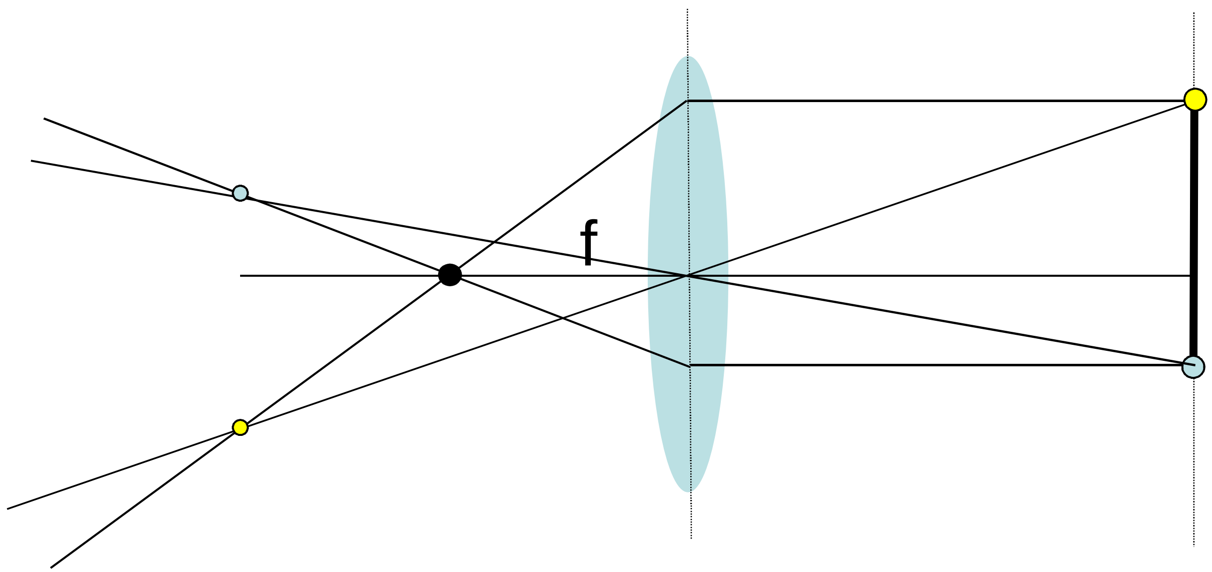
Example



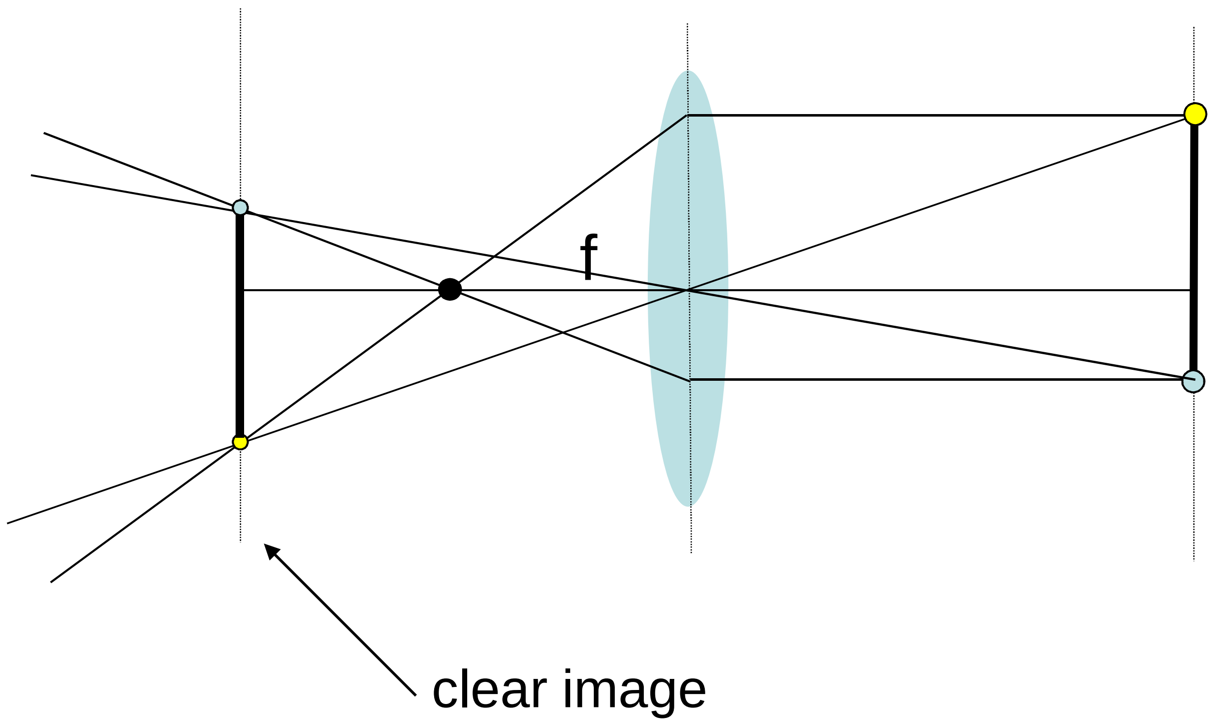
Example



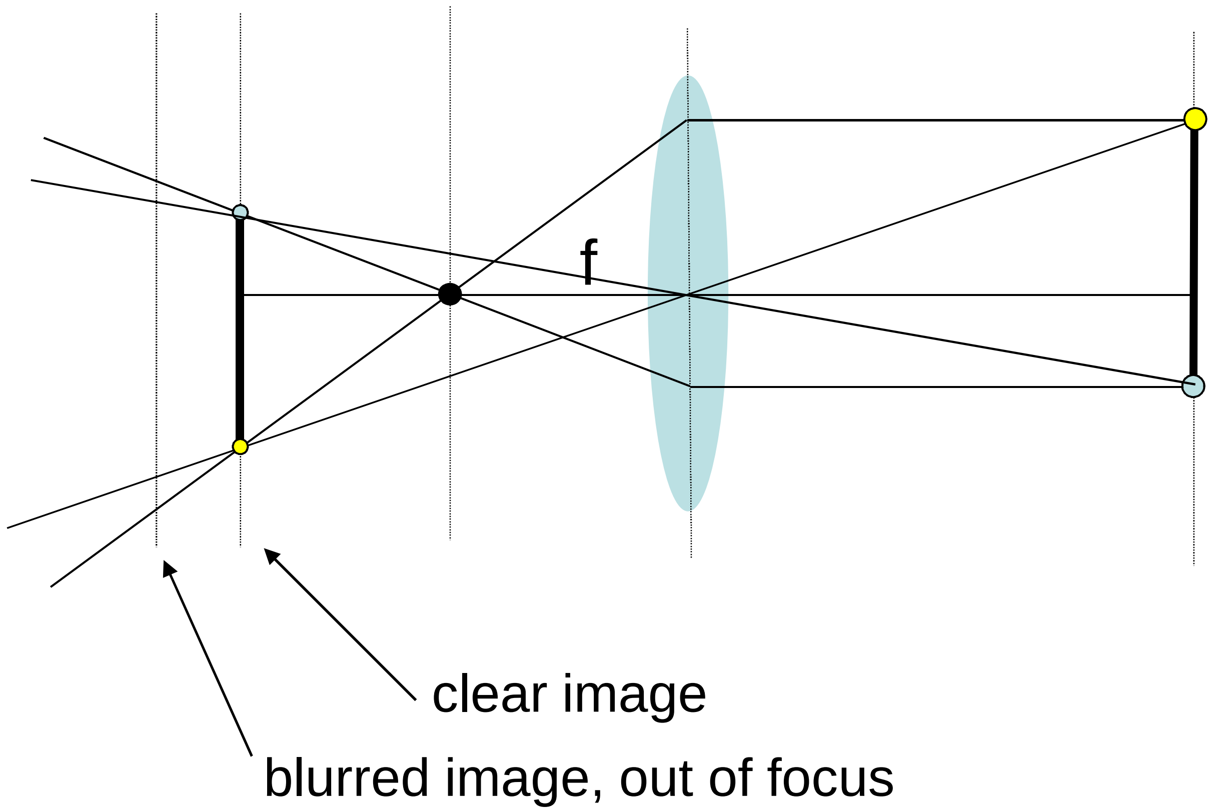
Example



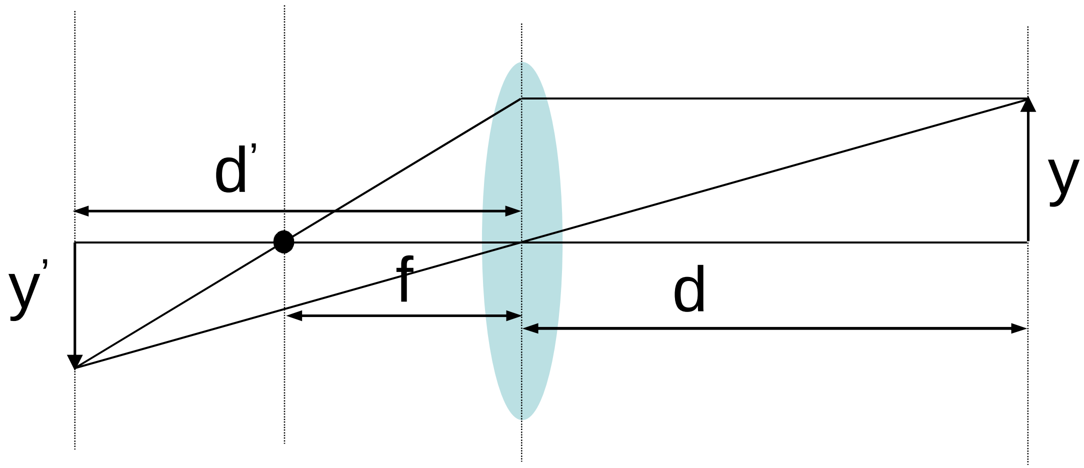
Example



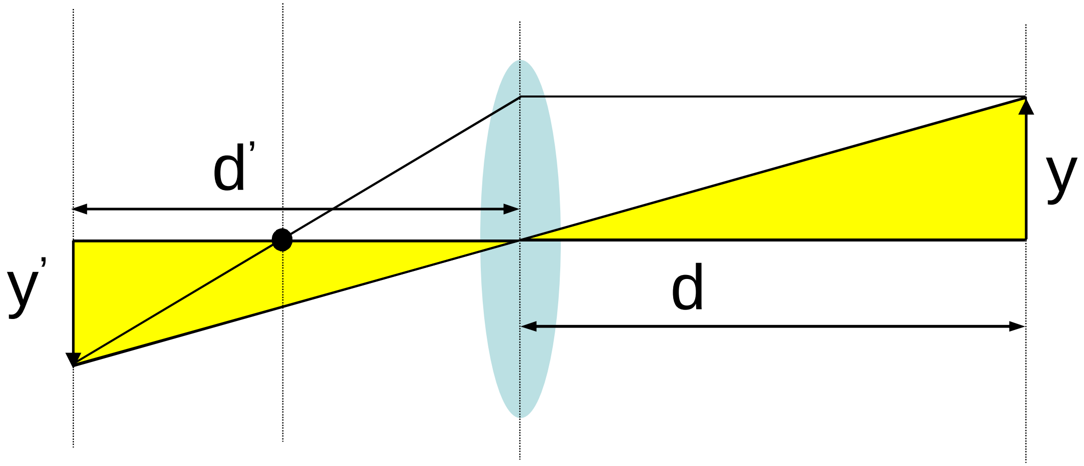
Example



Thin lens formula

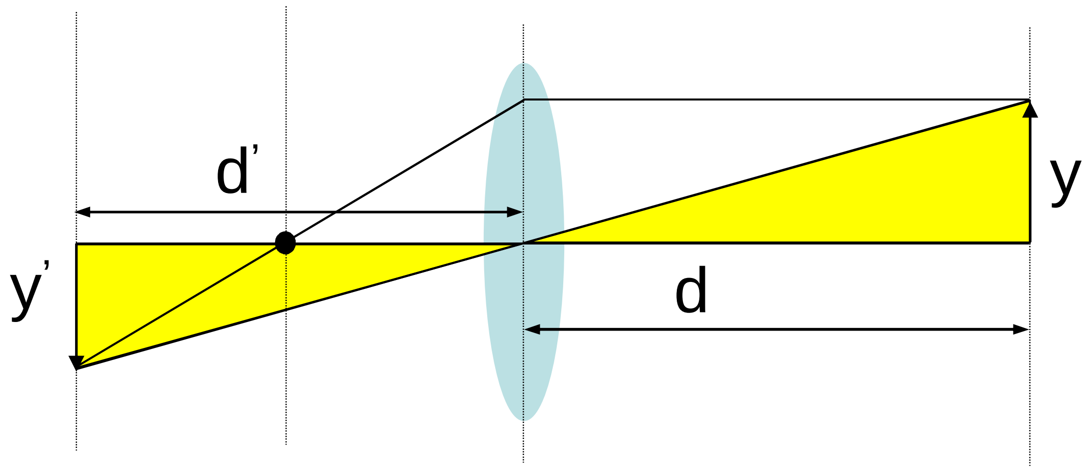


Thin lens formula

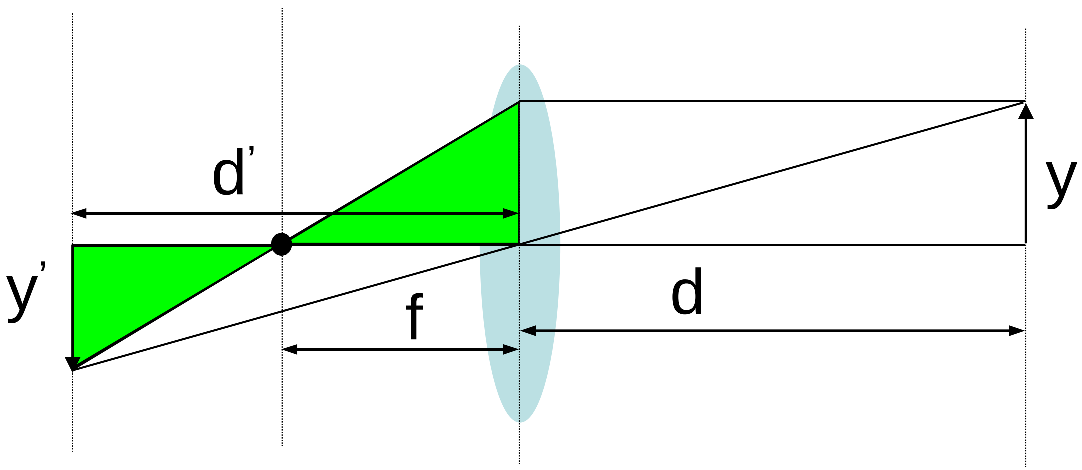


$$\frac{y'}{d'} = \frac{y}{d} \Rightarrow \frac{y'}{y} = \frac{d'}{d}$$

Thin lens formula



$$\frac{y'}{d'} = \frac{y}{d} \Rightarrow \frac{y'}{y} = \frac{d'}{d}$$

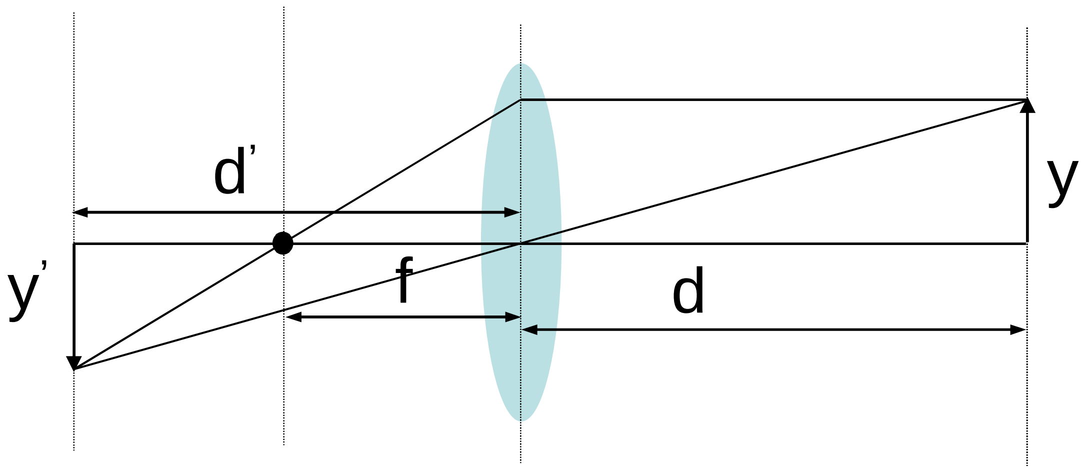


$$\frac{y'}{d'-f} = \frac{y}{f} \Rightarrow \frac{y'}{y} = \frac{d'-f}{f}$$

Thin lens formula

$$\begin{cases} \dfrac{y'}{y} = \dfrac{d'}{d} \\ \dfrac{y'}{y} = \dfrac{d'-f}{f} \end{cases}$$

$$\frac{d'}{d} = \frac{d'-f}{f} \Rightarrow \frac{d'}{d} = \frac{d'}{f} - 1 \Rightarrow \frac{1}{d} = \frac{1}{f} - \frac{1}{d'}$$

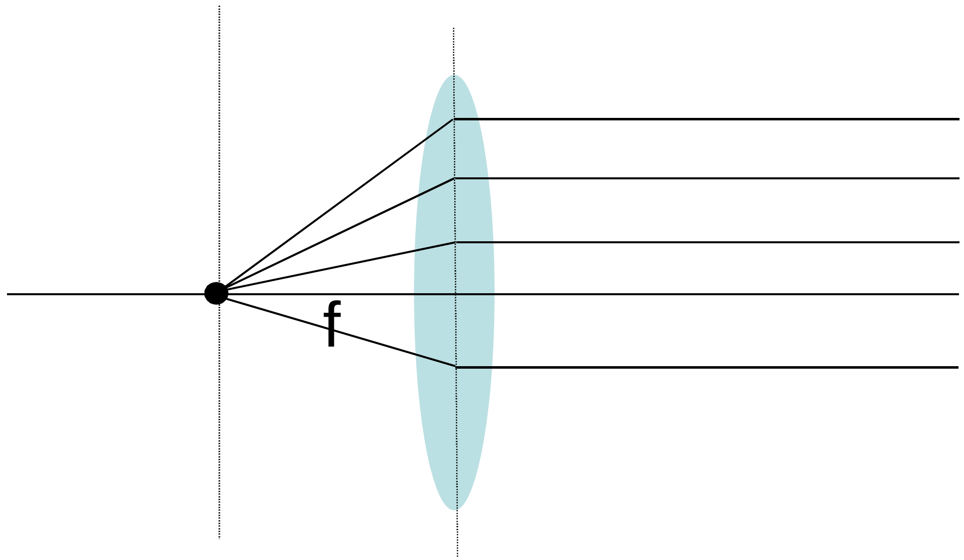


$$\frac{1}{d'} + \frac{1}{d} = \frac{1}{f}$$

Objects at infinity focus at *f*

$$\text{if } d \rightarrow \infty$$

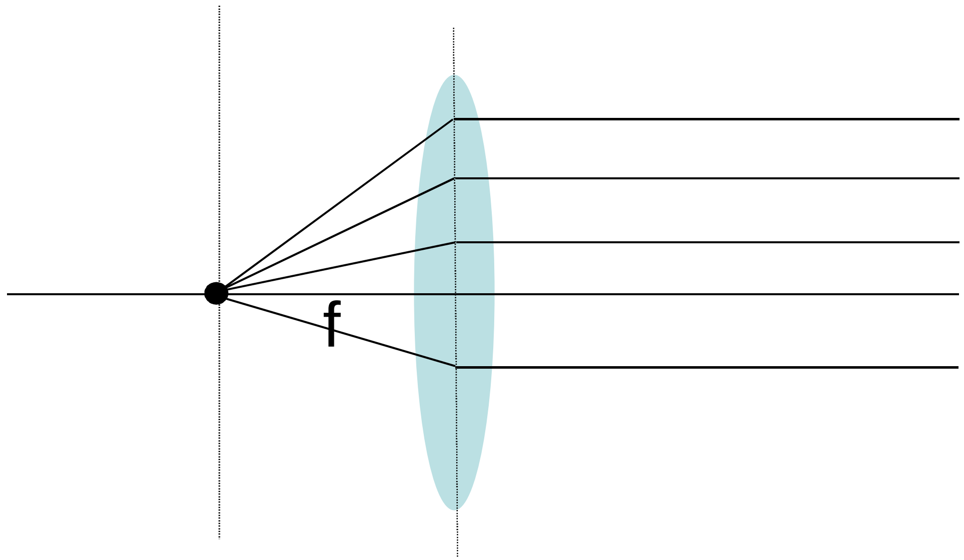
$$d' \rightarrow f$$



Objects at infinity focus at *f*

$$\text{if } d \rightarrow \infty$$

$$d' \rightarrow f$$

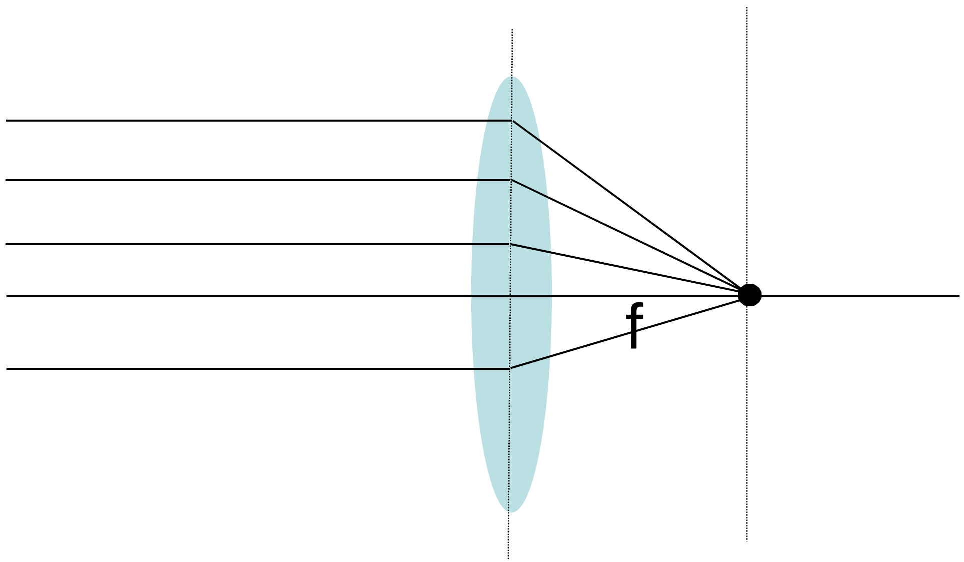


When the object gets closer, the focal plane moves away from *f*. At the limit:

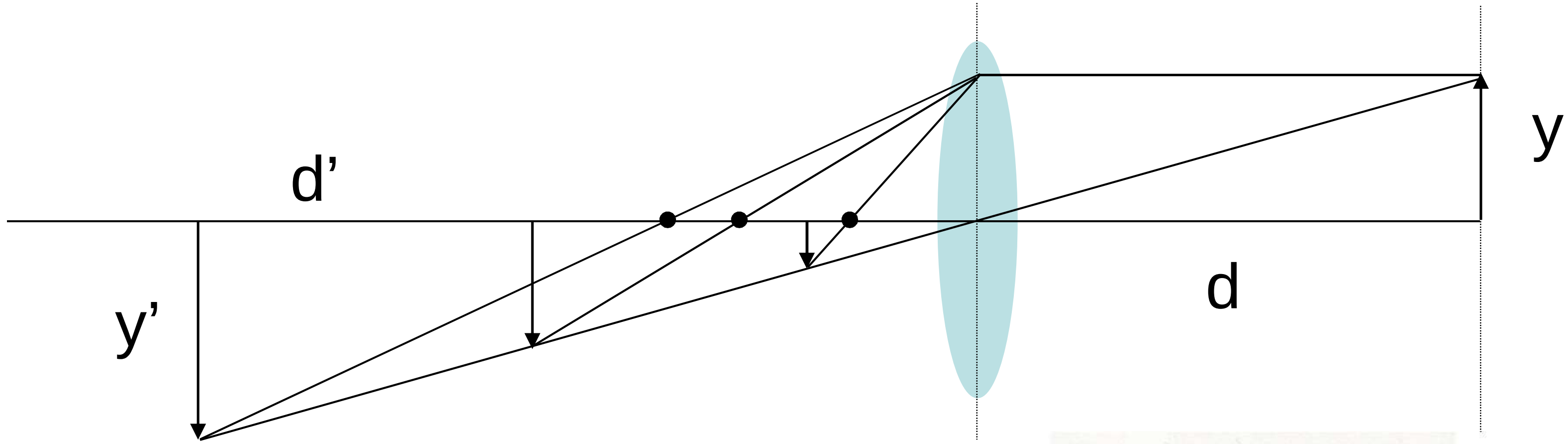
$$\text{if } d \rightarrow f$$

$$d' \rightarrow \infty$$

an object at distance *f* requires the focal plane to be at infinity



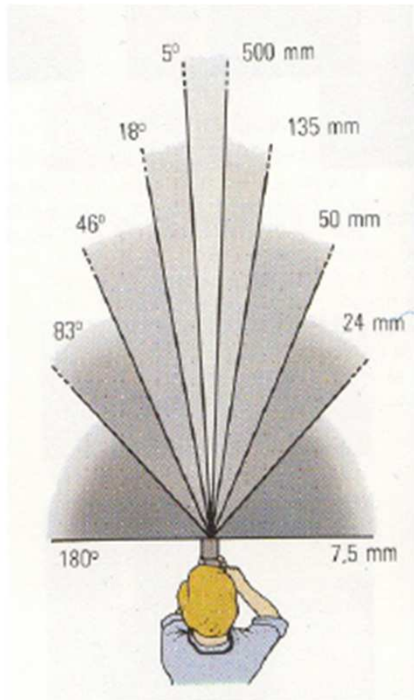
Effect of focal length on image size



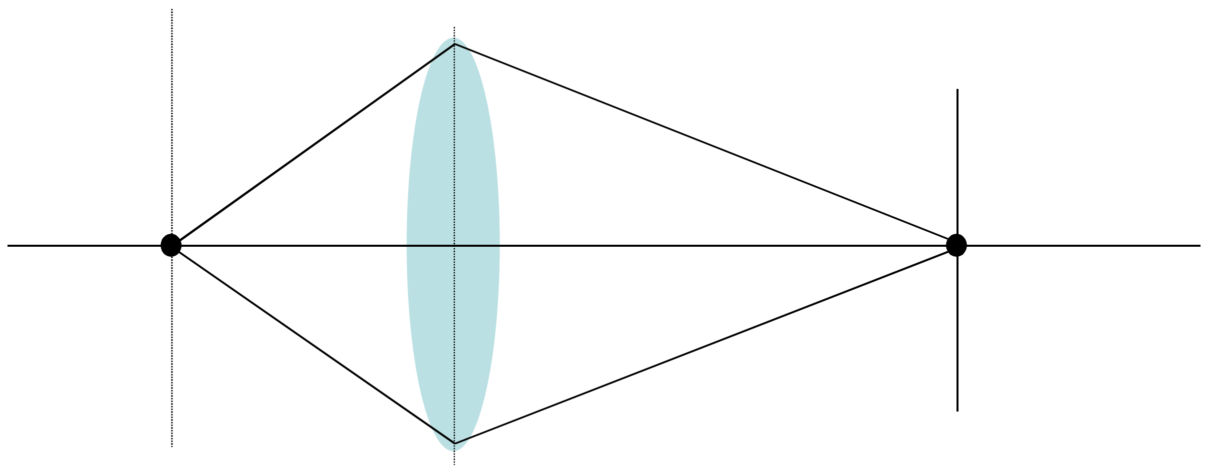
$$M = \frac{y'}{y} = \frac{d'}{d}$$

$$\frac{1}{f} = \frac{1}{d} + \frac{1}{d'} \Rightarrow M = \frac{f}{d-f} \quad , d > f$$

Effect of focal length on field of view

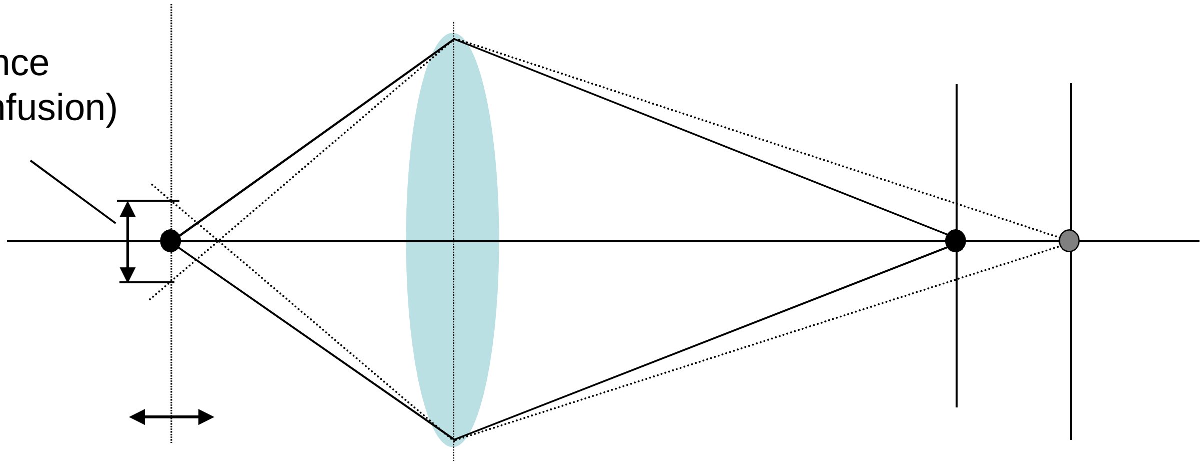


Depth of field (dof)

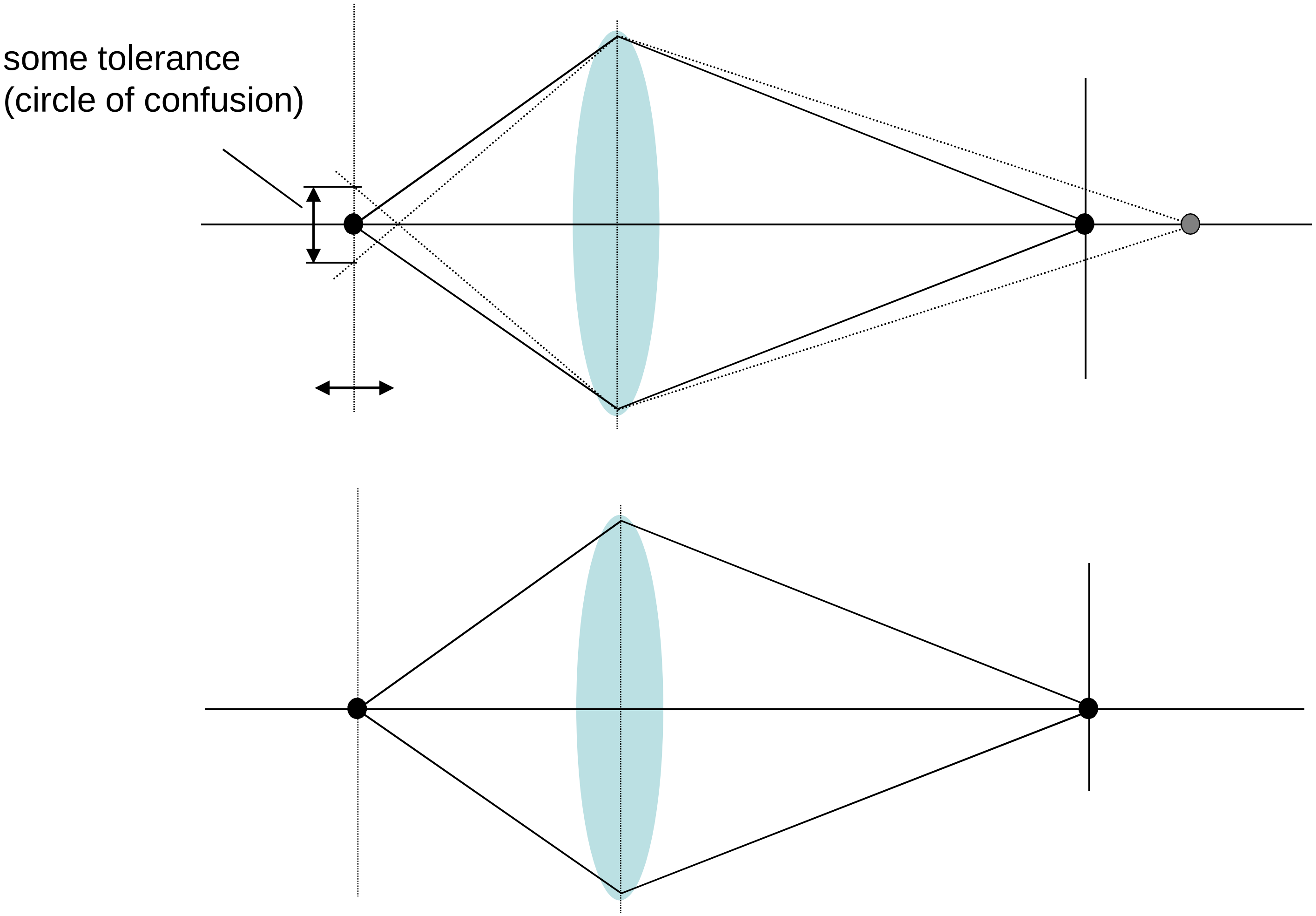


Depth of field (dof)

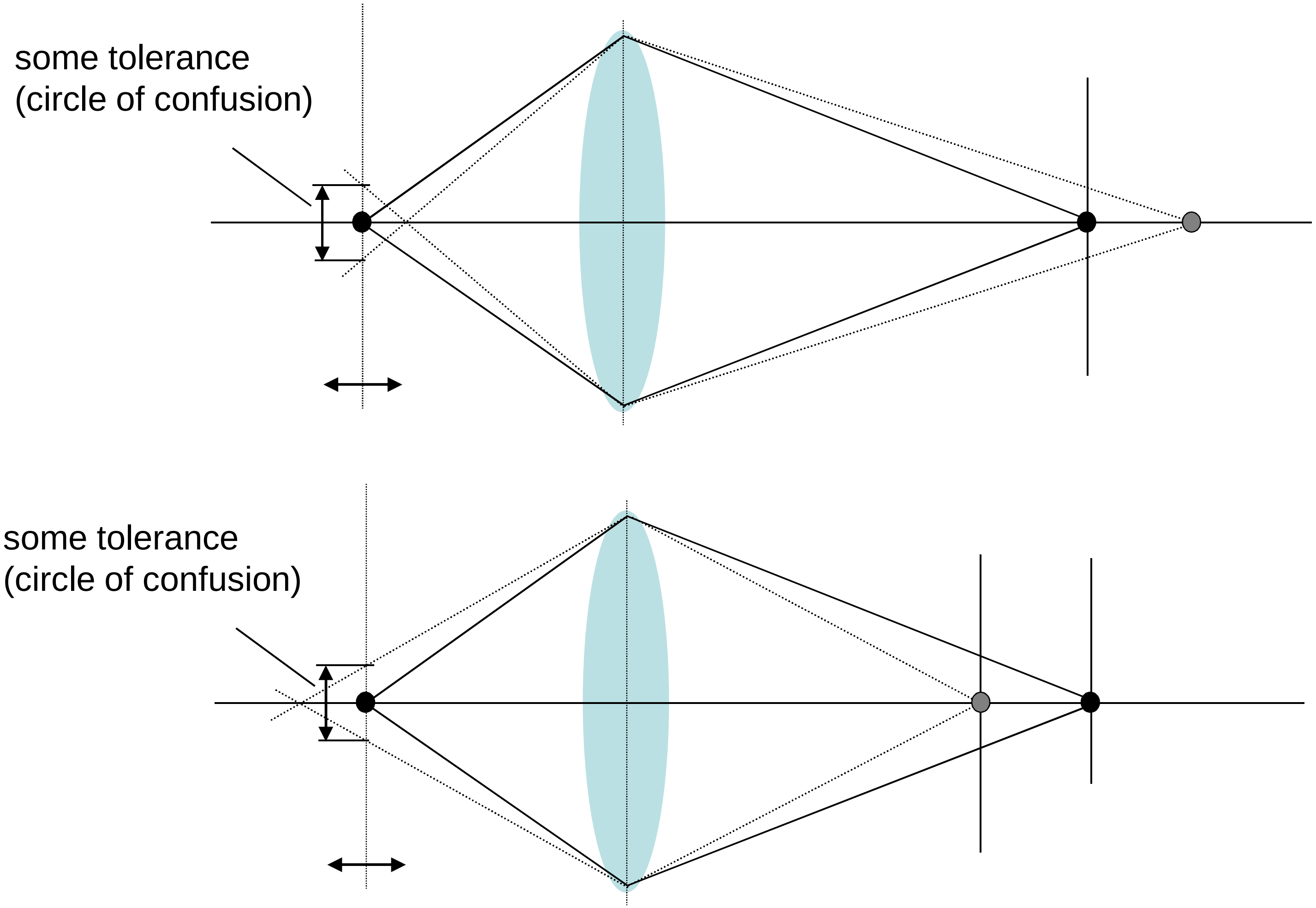
some tolerance
(circle of confusion)



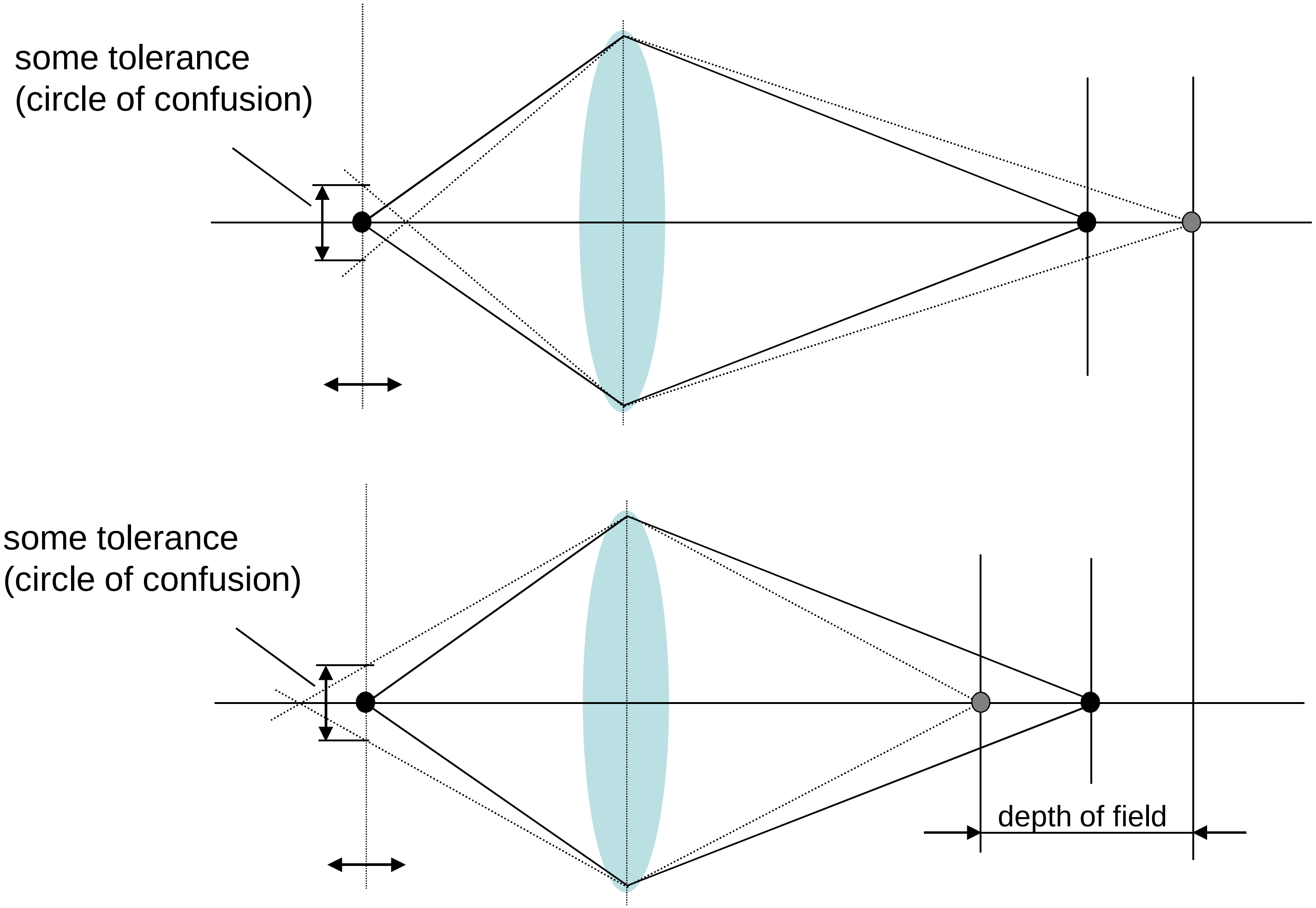
Depth of field (dof)



Depth of field (dof)



Depth of field (dof)



SINA – 10/11

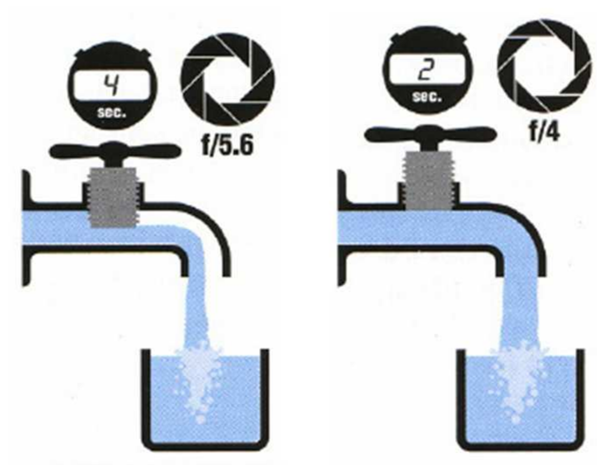
Getting the right exposure

- Shutter speed: how long the sensor is exposed to light, expressed in fractions of a second

  1/30 1/60 1/125 1/500 1/1000 ...

- Aperture: diaphragm controls how much light we allow through the lens (it is expressed as a fraction of focal length):

  (f/2.0, f/2.8, f/4, f/5.6, f/8 .. f/22)



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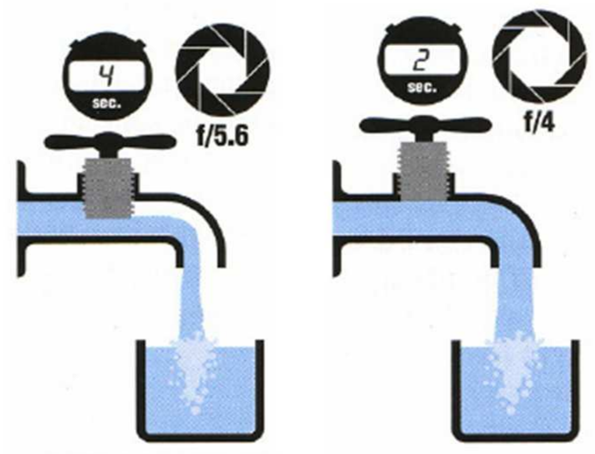
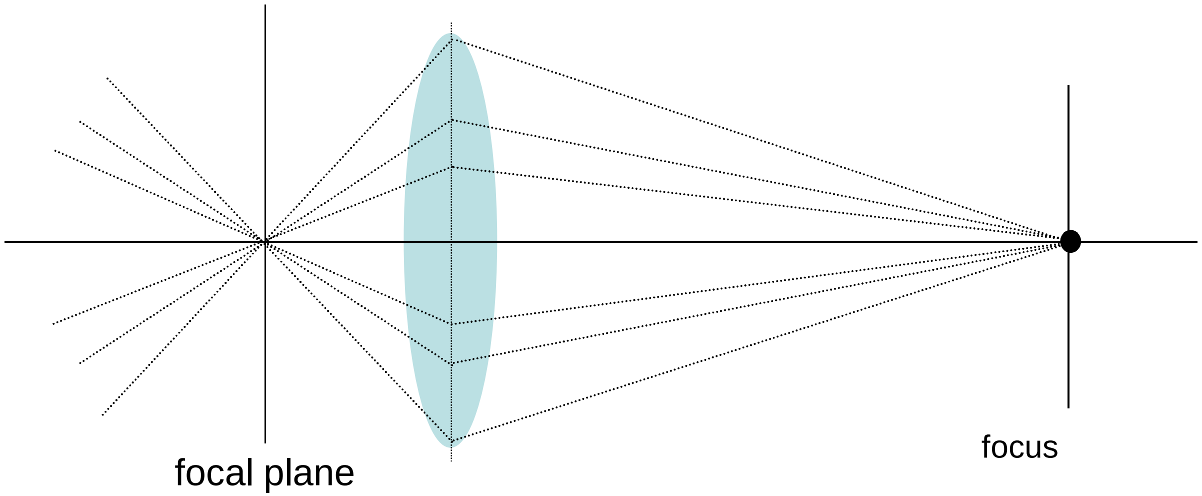
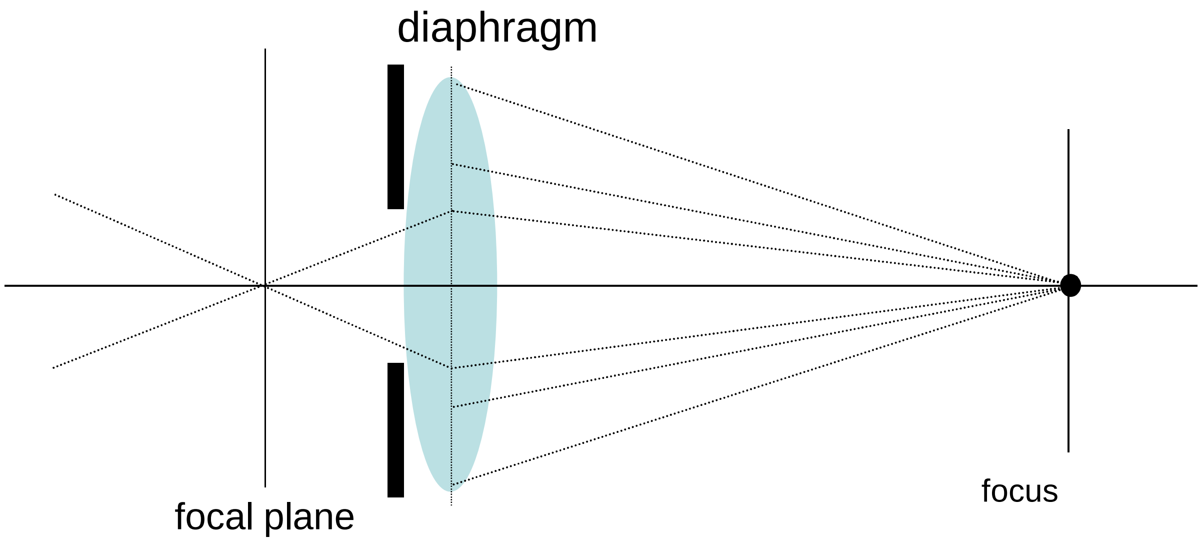
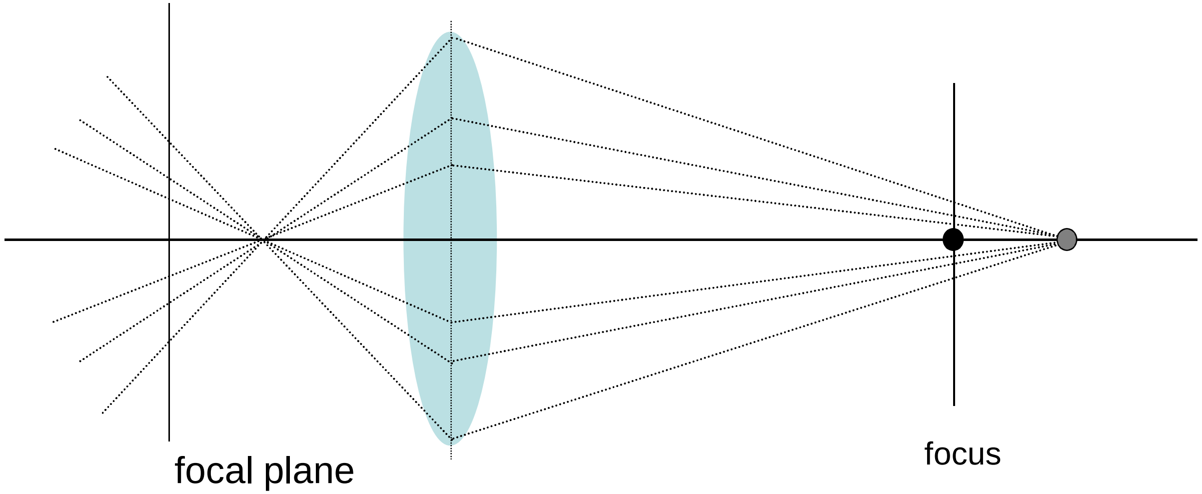
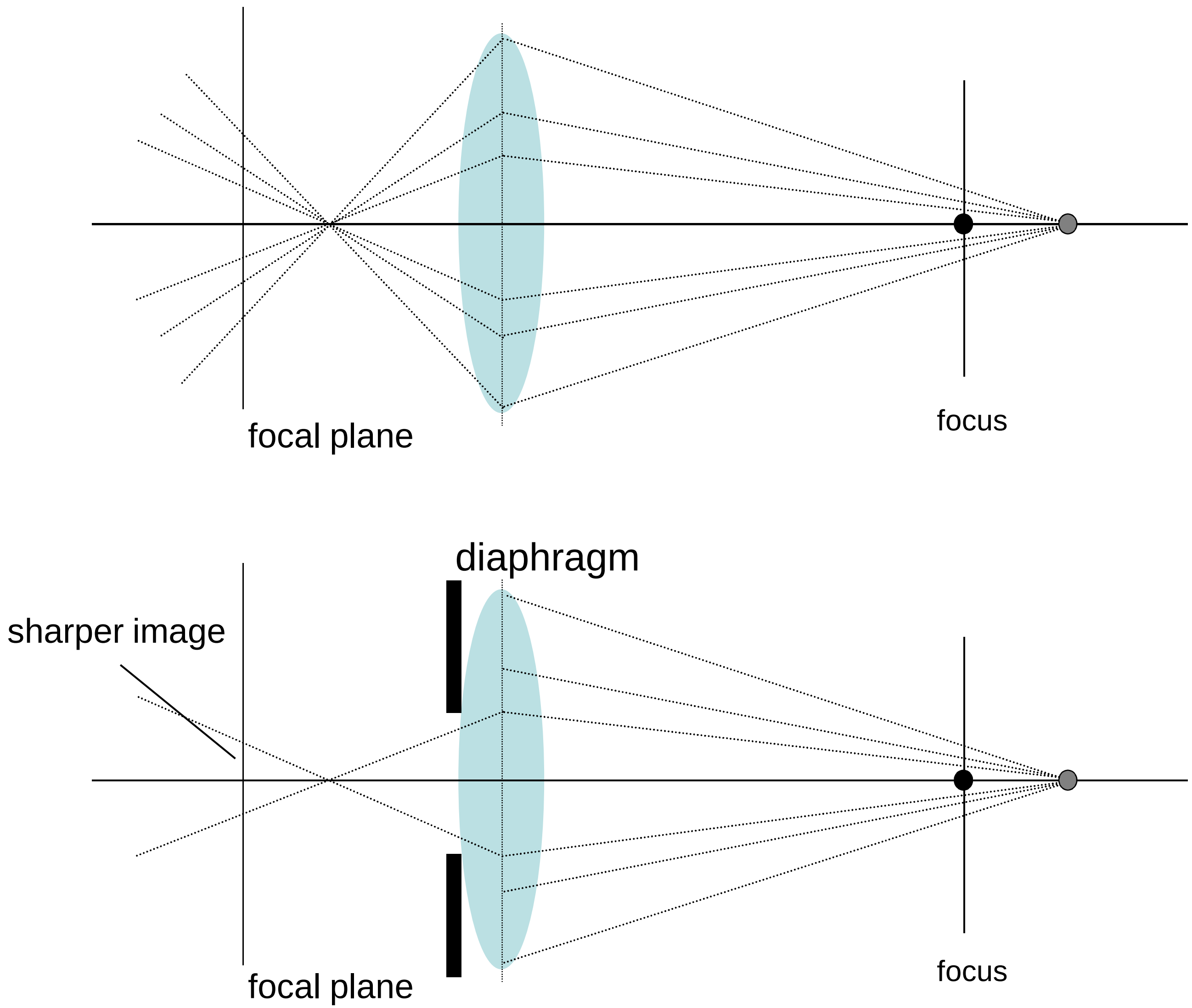


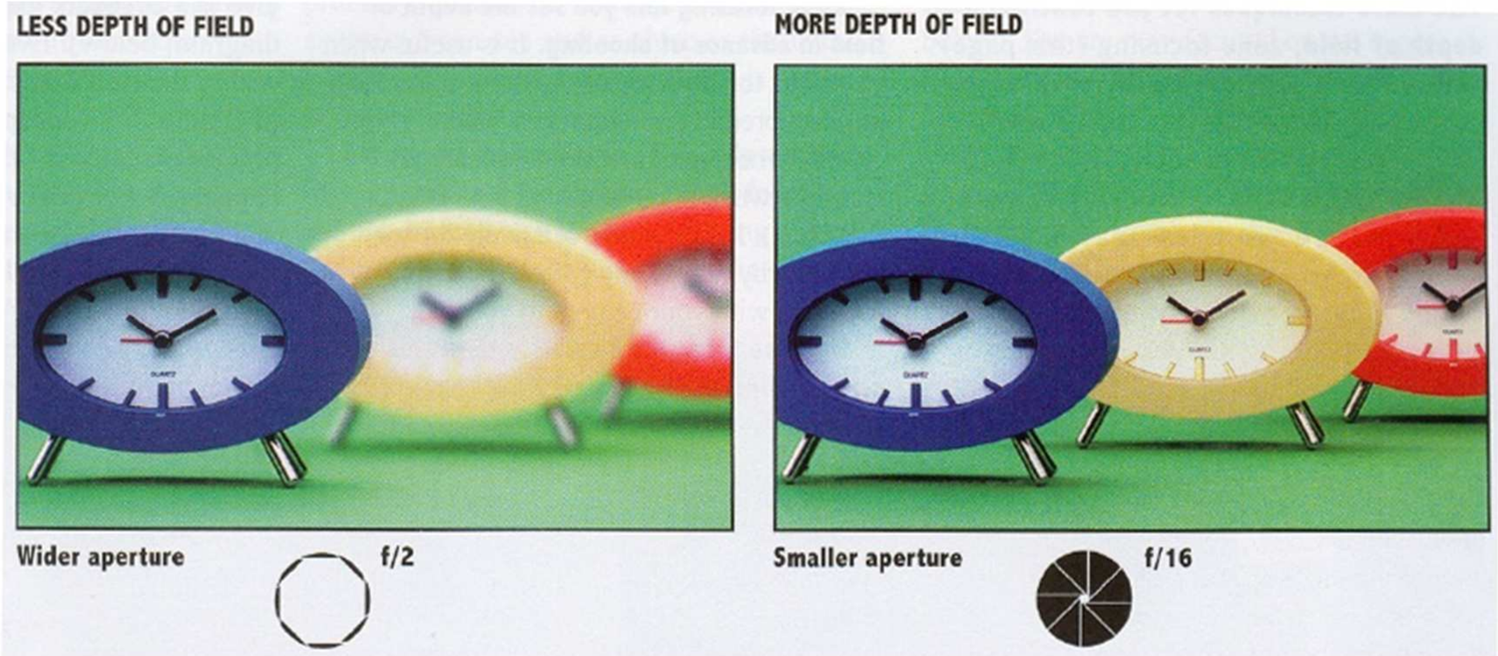

focal plane
focus


diaphragm
focal plane
focus

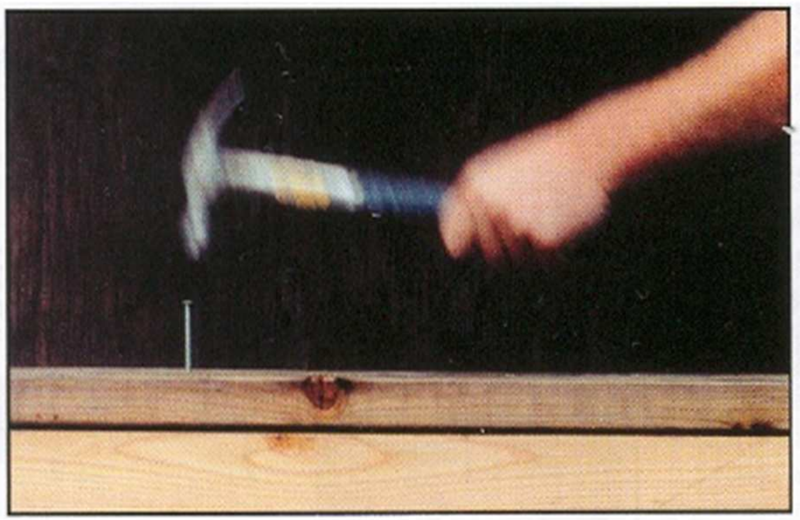

focal plane
focus


focal plane
focus
diaphragm
sharper image
focal plane
focus

Effect of aperture: depth of field



Effect of shutter speed: motion blur

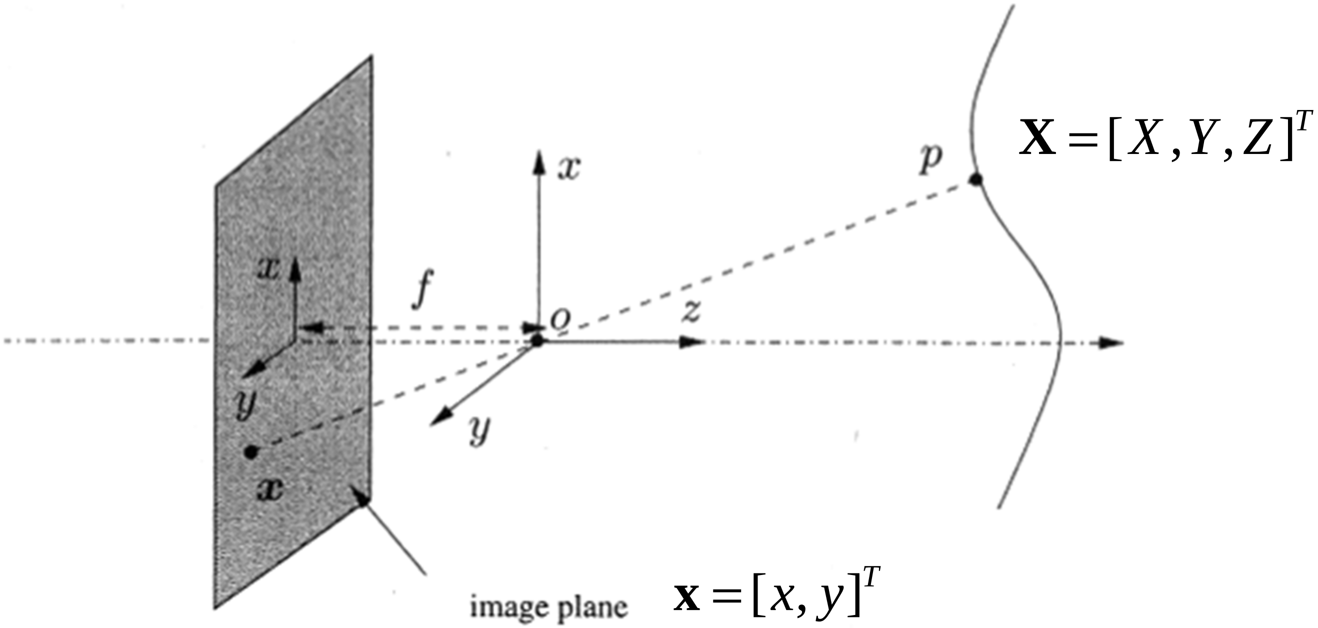


Slow shutter speed

Fast shutter speed

Image formation, camera model

Consider a pinhole camera, force all rays to go through the optical center

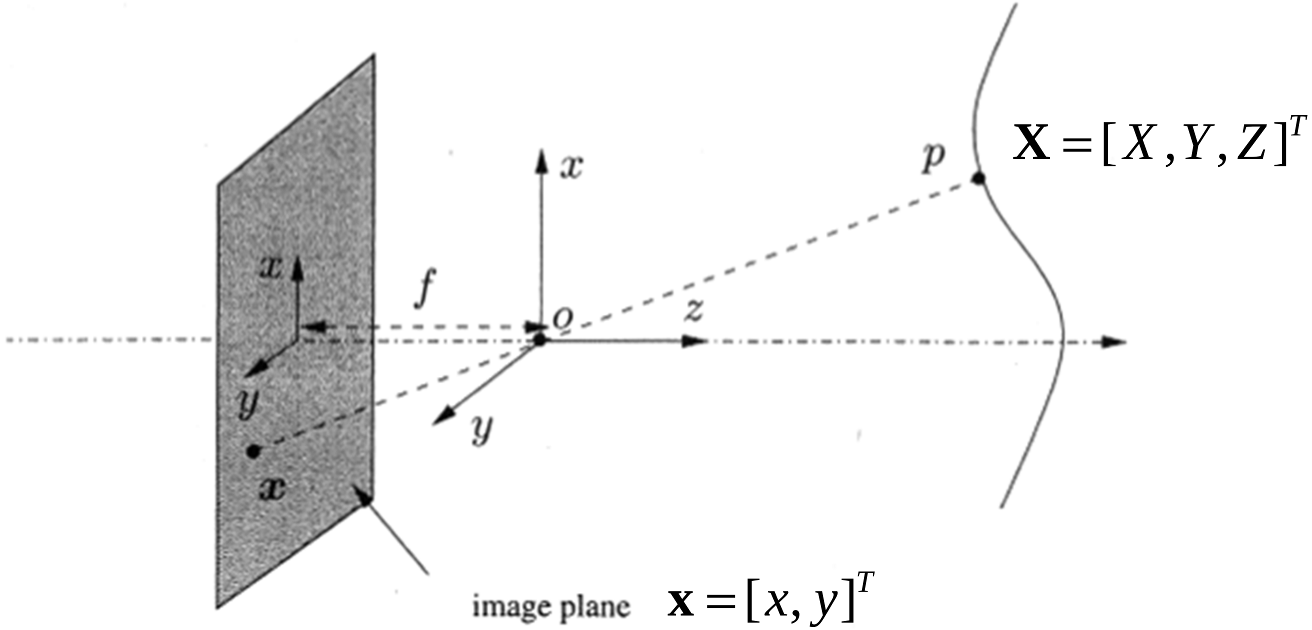


$$\begin{cases} x = \lambda X \\ y = \lambda Y \\ z = \lambda Z \end{cases}$$

See: Forsyth and Ponce, Computer Vision a Modern Approach

Image formation, camera model

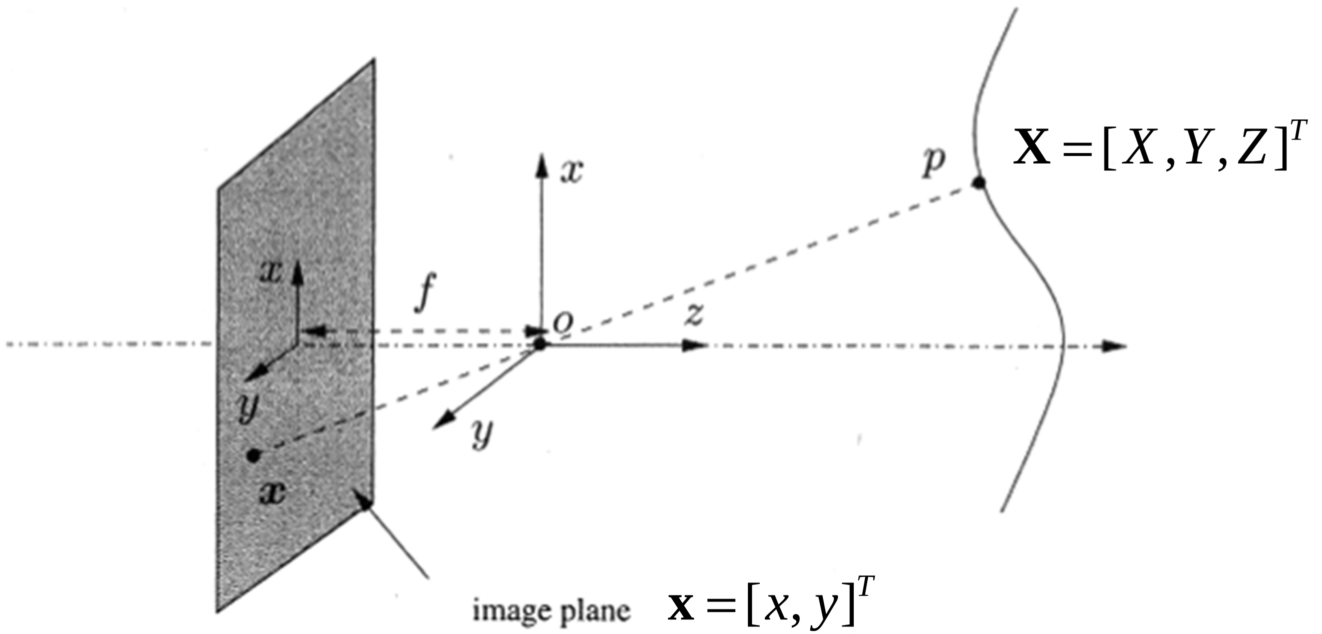
Consider a pinhole camera, force all rays to go through the optical center



$$\begin{cases} x = \lambda X \\ y = \lambda Y \\ z = \lambda Z \end{cases} \Rightarrow \begin{cases} x = \lambda X \\ y = \lambda Y \\ z = -f \Rightarrow \lambda = -f / Z \end{cases}$$

Image formation, camera model

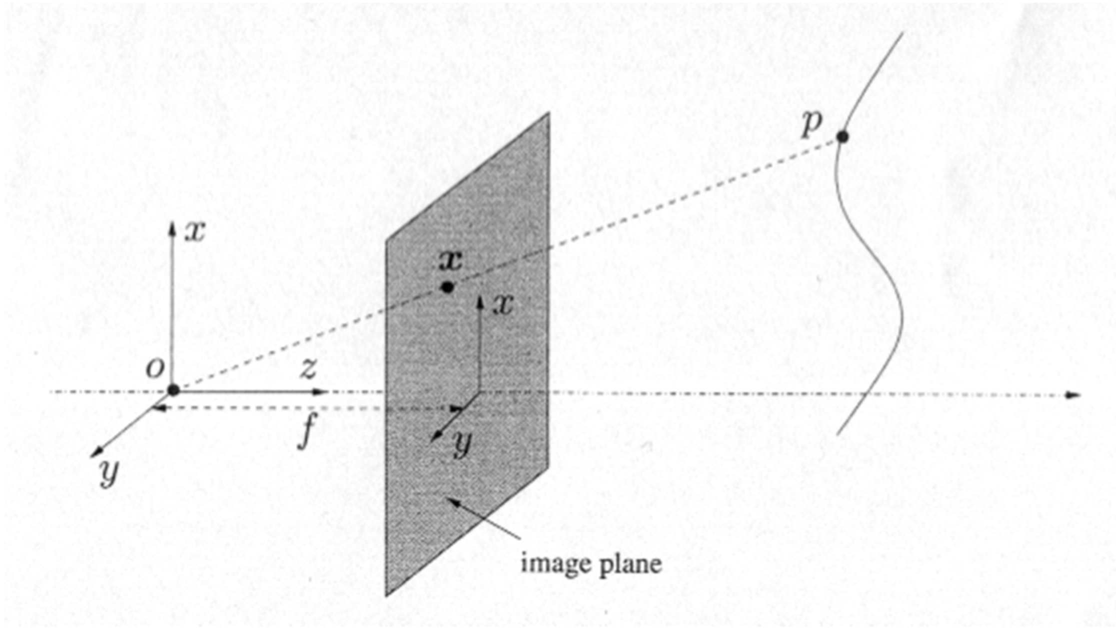
Consider a pinhole camera, force all rays to go through the optical center



$$\begin{cases} x = \lambda X \\ y = \lambda Y \\ z = \lambda Z \end{cases} \Rightarrow \begin{cases} x = \lambda X \\ y = \lambda Y \\ z = -f \Rightarrow \lambda = -f / Z \end{cases} \Rightarrow x = -f \frac{X}{Z}, y = -f \frac{Y}{Z}$$

ideal pinhole camera model

Often we flip the image (-x,-y) → (x,y), which is equivalent to placing the image plane in front of the optical center:



$$x = f\,\frac{X}{Z}, \quad y = f\,\frac{Y}{Z}$$

Note: any point on the line through *o* and *p* projects onto the same coordinates *(x,y)*

- Consider a generic point *p* with coordinates $\mathbf{X}_0$=[$X_{0,}$ Y_0 Z_0] relative to the world reference frame
- The coordinates **X**=[X, Y Z] of *p* relative to the camera frame are given by the rigid body transformation:

$$\mathbf{X} = \mathbf{R} \cdot \mathbf{X_0} + \mathbf{T}$$

$$\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \\ 1 \end{bmatrix}$$

homogeneous representation

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{f}{Z} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{f}{Z} \begin{bmatrix} X \\ Y \end{bmatrix} \Rightarrow Z \begin{bmatrix} x \\ y \end{bmatrix} = f \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{f}{Z} \begin{bmatrix} X \\ Y \end{bmatrix} \Rightarrow Z \begin{bmatrix} x \\ y \end{bmatrix} = f \begin{bmatrix} X \\ Y \end{bmatrix} \Rightarrow Z \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{f}{Z} \begin{bmatrix} X \\ Y \end{bmatrix} \Rightarrow Z \begin{bmatrix} x \\ y \end{bmatrix} = f \begin{bmatrix} X \\ Y \end{bmatrix} \Rightarrow Z \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$Z \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{f}{Z} \begin{bmatrix} X \\ Y \end{bmatrix} \Rightarrow Z \begin{bmatrix} x \\ y \end{bmatrix} = f \begin{bmatrix} X \\ Y \end{bmatrix} \Rightarrow Z \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$Z \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

replace Z with an arbitrary positive scalar

$$\lambda \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \\ 1 \end{bmatrix}$$

consider a point in the world reference frame

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{f}{Z}\begin{bmatrix} X \\ Y \end{bmatrix} \Rightarrow Z\begin{bmatrix} x \\ y \end{bmatrix} = f\begin{bmatrix} X \\ Y \end{bmatrix} \Rightarrow Z\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$Z \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

replace Z with an arbitrary positive scalar

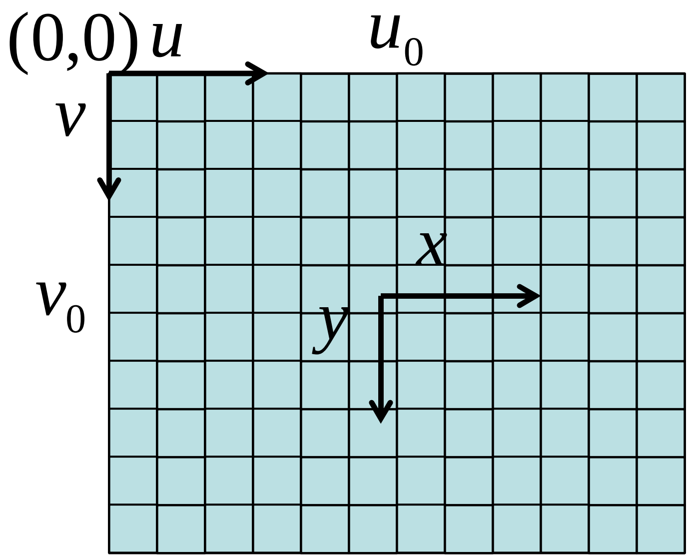
$$\lambda \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \\ 1 \end{bmatrix}$$

consider a point in the world reference frame

$$\boxed{\lambda \cdot \mathbf{x} = K_f M_0 g \mathbf{X}_0}$$

geometric model for *an ideal camera*

Intrinsic parameters



Pixel size:

$$\frac{1}{k} \times \frac{1}{l} \quad \text{units: [pixels/m]}$$

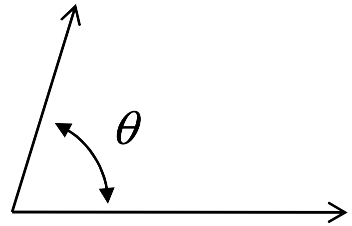
$$u = k \cdot x + u_0$$

$$v = l \cdot x + v_0$$

$$\mathbf{p} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} k & 0 & u_0 \\ 0 & l & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- If pixels are not rectangular, a more general form of matrix is considered:

$$\mathbf{p} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} k & s_\theta & u_0 \\ 0 & l & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



where s_θ is called *skew factor*

- If pixels are not rectangular, a more general form of matrix is considered:

$$\mathbf{p} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} k & s_\theta & u_0 \\ 0 & l & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

where s_θ is called *skew factor*

- A more realistic model of a transformation between homogeneous coordinates of a 3D point relative to the world reference frame and its image in terms of pixels:

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} k & s_\theta & u_0 \\ 0 & l & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{T} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \\ 1 \end{bmatrix}$$

$$K = K_s \cdot K_f = \begin{bmatrix} kf & fs_\theta & u_0 \\ 0 & lf & v_0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & \gamma_\theta & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

- To summarize:

extrinsic parameters

$$\lambda \mathbf{p} = K M_0 g \mathbf{X}_0 = M \mathbf{P}$$

$K = K_s K_f$ *intrinsic parameters*

- Intrinsic and extrinsic parameters can be estimated with a general technique called "*camera calibration*" (see for example: *R.Y. Tsai 1986)*

Projection matrix: characterization

- The projection matrix can be written explicitly as a function of its five intrinsic parameters and the six extrinsic ones (we skip the details):

$$M = \begin{pmatrix} \alpha \mathbf{r}_1^T - \gamma_\theta \mathbf{r}_2^T + u_0 \mathbf{r}_3^T & \alpha t_x - \gamma_\theta t_y + u_0 t_z \\ \beta \mathbf{r}_2^T + v_0 \mathbf{r}_3^T & \beta t_y + v_0 t_z \\ \mathbf{r}_3^T & t_z \end{pmatrix}$$

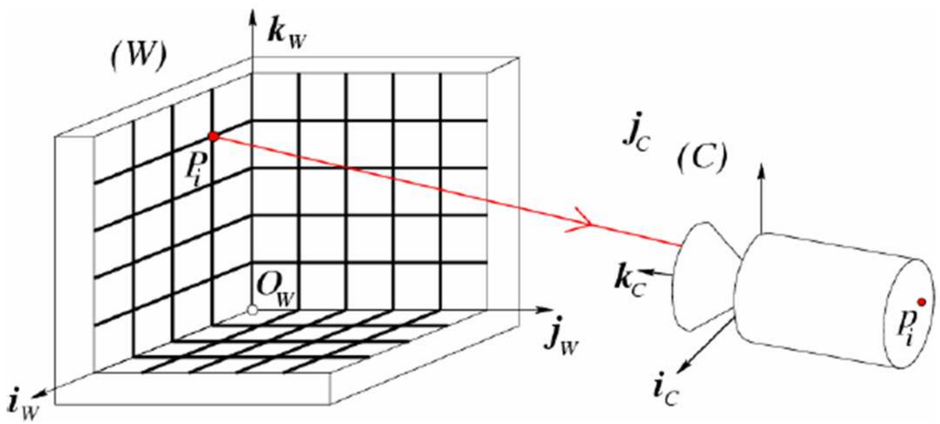
$\mathbf{r}_1^T, \mathbf{r}_2^T, \mathbf{r}_3^T$ denote the three rows of the matrix $\mathbf{R}$

t_x, t_y, t_z are the coordinates of the vector $\mathbf{T}$

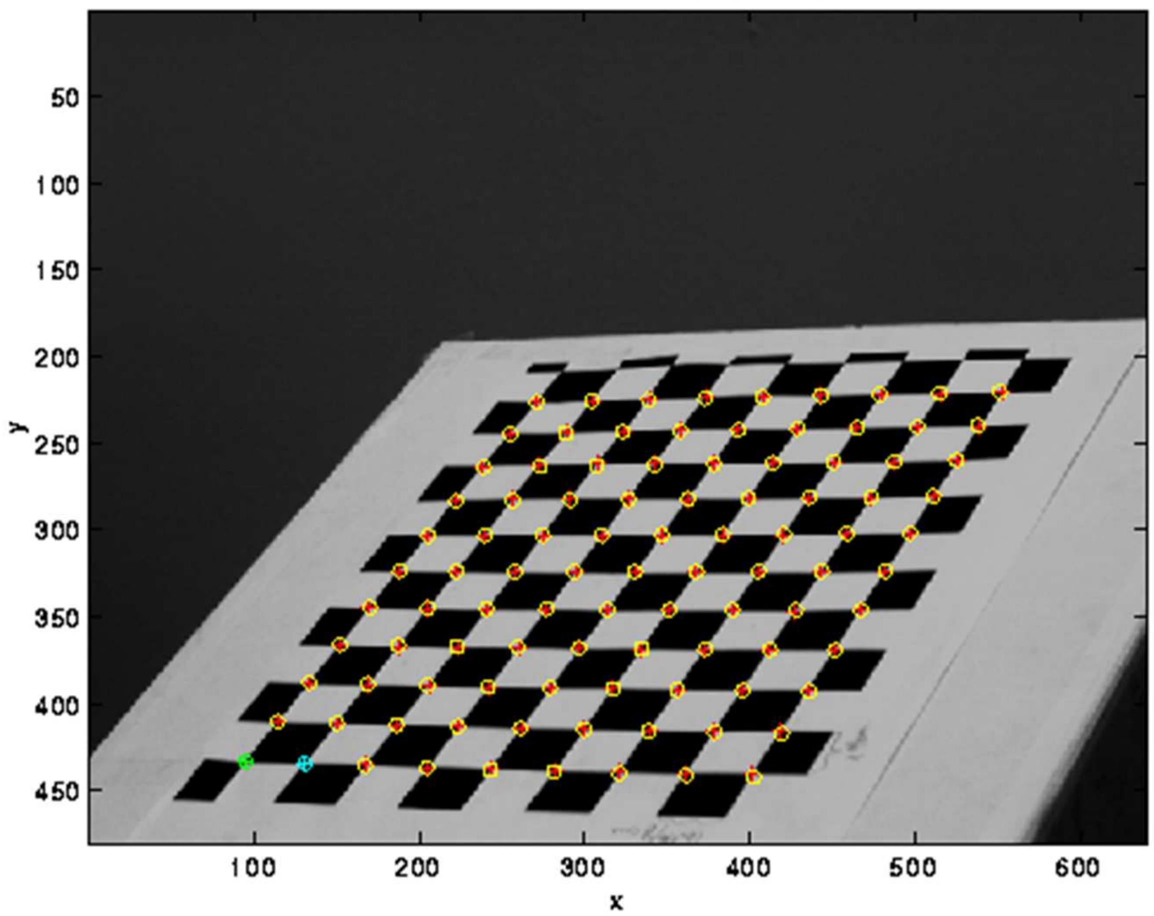
- *M* is a 3x4 matrix
- Given the structure of **R**, *M* has 11 degrees of freedom: 5 intrinsic parameters + 6 extrinsic ones (3 for rotation and 3 for translation)

Geometric Camera Calibration (introduction)

- We assume that the camera observes a set of features such as points or lines with known positions in a fixed world coordinate system
- These points can be localized automatically or manually
- Goal: derive the intrinsic and extrinsic parameters of the camera
- Allow associating with any image point a well-defined ray passing through the point and the camera's optical center



Calibration Pattern with the projected points



Linear Approach

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = M \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \Leftrightarrow \lambda \mathbf{p} = M\mathbf{P}$$

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{13} \\ m_{21} & m_{22} & m_{23} & m_{23} \\ m_{31} & m_{32} & m_{33} & m_{31} \end{bmatrix} = \begin{bmatrix} \mathbf{m}_1^T \\ \mathbf{m}_2^T \\ \mathbf{m}_3^T \end{bmatrix}$$

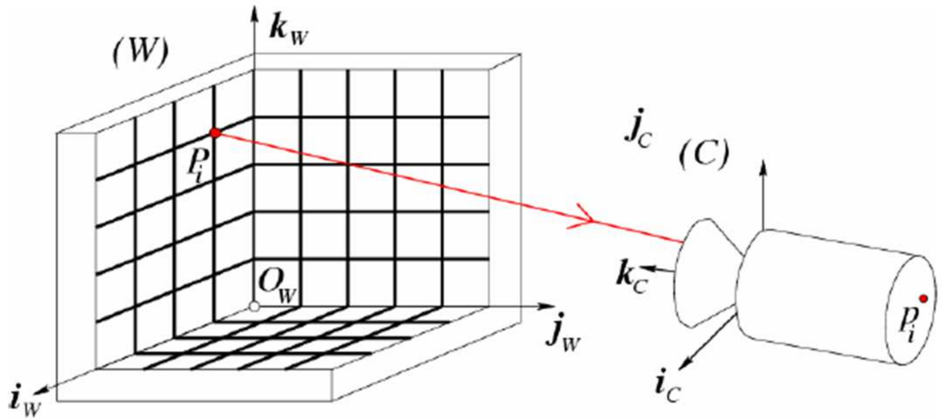
$$\mathbf{m}_i \stackrel{def}{=} \begin{bmatrix} m_{i,1} \\ m_{i,2} \\ m_{i,3} \end{bmatrix}$$

$$\mathbf{m}_1^T, \mathbf{m}_2^T, \mathbf{m}_3^T \text{ rows of } M$$

$$\lambda = \mathbf{m}_3 \mathbf{P} \Rightarrow \begin{cases} u = \dfrac{\mathbf{m_1} \cdot \mathbf{P}}{\mathbf{m}_3 \cdot \mathbf{P}} \\ v = \dfrac{\mathbf{m}_2 \cdot \mathbf{P}}{\mathbf{m}_3 \cdot \mathbf{P}} \end{cases}$$

- *M* is the projection matrix it contains extrinsic and intrinsic parameters of the camera
- Algorithms for camera calibration usually consists in two steps:
  1. *Estimate M*
  2. *Reconstruct* the camera parameters from *M*

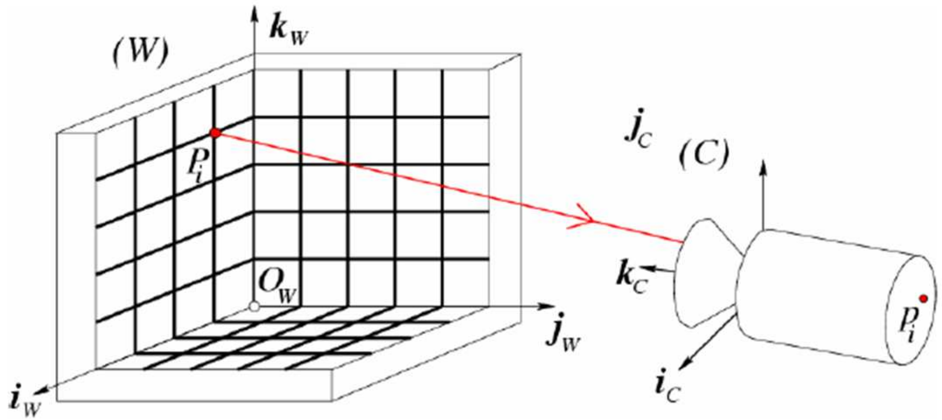
Linear Approach



Consider a set of *n* points with *known* position P_i, and projection u_i,v_i

$$\begin{cases} u_i = \dfrac{\mathbf{m}_1 \cdot \mathbf{P}_i}{\mathbf{m}_3 \cdot \mathbf{P}_i} \\[2ex] v_i = \dfrac{\mathbf{m}_2 \cdot \mathbf{P}_i}{\mathbf{m}_3 \cdot \mathbf{P}_i} \end{cases}$$

Linear Approach



Consider a set of *n* points with *known* position P_i, and projection u_i,v_i

For each point *i* we get two equations

$$\begin{cases} u_i = \dfrac{\mathbf{m_1} \cdot \mathbf{P}_i}{\mathbf{m}_3 \cdot \mathbf{P}_i} \\ v_i = \dfrac{\mathbf{m}_2 \cdot \mathbf{P}_i}{\mathbf{m}_3 \cdot \mathbf{P}_i} \end{cases} \Rightarrow \begin{cases} (\mathbf{m_1} - u_i \mathbf{m}_3) \cdot \mathbf{P}_i = 0 \\ (\mathbf{m}_2 - v_i \mathbf{m}_3) \cdot \mathbf{P}_i = 0 \end{cases}$$

We organize the equations in matrix form:

$$\begin{cases} u_i \mathbf{m}_3 \mathbf{P}_i = \mathbf{m}_1 \mathbf{P}_i \\ v_i \mathbf{m}_3 \mathbf{P}_i = \mathbf{m}_2 \mathbf{P}_i \end{cases}$$

$$\begin{cases} u_i m_{31} X_i + u_i m_{32} Y_i + u_i m_{33} Z_i + u_i m_{34} = m_{11} X_i + m_{12} Y_i + m_{13} Z_i + m_{14} \\ v_i m_{31} X_i + v_i m_{32} Y_i + v_i m_{33} Z_i + v_i m_{34} = m_{21} X_i + m_{22} Y_i + m_{23} Z_i + m_{24} \end{cases}$$

$$\begin{cases} -(m_{11} X_i + m_{12} Y_i + m_{13} Z_i + m_{14}) + u_i m_{31} X_i + u_i m_{32} Y_i + u_i m_{33} Z_i + u_i m_{34} = 0 \\ -(m_{21} X_i + m_{22} Y_i + m_{23} Z_i + m_{24}) + v_i m_{31} X_i + v_i m_{32} Y_i + v_i m_{33} Z_i + v_i m_{34} = 0 \end{cases}$$

in matrix form:

$$\begin{bmatrix} -X_i & -Y_i & -Z_i & -1 & 0 & 0 & 0 & 0 & u_i X_i & u_i Y_i & u_i Z_i & u_i \\ 0 & 0 & 0 & 0 & -X_i & -X_i & -X_i & -1 & v_i X_i & v_i Y_i & u_i Z_i & v_i \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{12} \\ m_{13} \\ m_{14} \\ \dots \\ m_{33} \\ m_{34} \end{bmatrix} = 0$$

Collecting *n* points leads to 2n equations:

$$
\begin{bmatrix}
-X_0 & -Y_0 & -Z_0 & -1 & 0 & 0 & 0 & 0 & u_0X_0 & u_0Y_0 & u_0Z_0 & u_0 \\
0 & 0 & 0 & 0 & -X_0 & -X_0 & -X_0 & -1 & v_0X_0 & v_0Y_0 & u_0Z_0 & v_0 \\
-X_1 & -Y_1 & -Z_1 & -1 & 0 & 0 & 0 & 0 & u_1X_1 & u_1Y_1 & u_1Z_1 & u_1 \\
0 & 0 & 0 & 0 & -X_1 & -X_1 & -X_1 & -1 & v_1X_1 & v_1Y_1 & v_1Z_1 & v_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-X_{n-1} & -Y_{n-1} & -Z_{n-1} & -1 & 0 & 0 & 0 & 0 & u_{n-1}X_{n-1} & u_{n-1}Y_{n-1} & u_{n-1}Z_{n-1} & u_{n-1} \\
0 & 0 & 0 & 0 & -X_{n-1} & -Y_{n-1} & -Z_{n-1} & -1 & v_{n-1}X_{n-1} & v_{n-1}Y_{n-1} & v_{n-1}Z_{n-1} & v_{n-1}
\end{bmatrix}
\begin{bmatrix} m_{11} \\ m_{12} \\ m_{13} \\ m_{14} \\ \ldots \\ m_{33} \\ m_{34} \end{bmatrix} = \mathbf{0}
$$

in compact form:

$$\mathbf{W} \cdot \mathbf{m} = \mathbf{0}$$

- m is 12x1 (M has 12 coefficients)
- W is a 2nx12 matrix
- When *n* is large (>6) *homogeneous least-squares* can be used to determine **m** (and the projection matrix **M**), the solution is the eigenvector of W^TW such that |**m**|=1 (more on next slide)

Homogeneous Least-squares

- Consider the following problem

$$\begin{cases} u_{11}x_1 + u_{12}x_2 + ... + u_{1q}x_q = 0 \\ u_{21}x_1 + u_{22}x_2 + ... + u_{2q}x_q = 0 \\ ... \\ u_{p1}x_1 + u_{p2}x_2 + ... + u_{pq}x_q = 0 \end{cases} \Leftrightarrow \mathbf{U} \cdot \mathbf{x} = 0$$

- The following cases:
  - p=q and U is non singular, unique solution x=0
  - p>=q non zero solutions exist when U is singular with rank<q
  - To find a non trivial solution we set the additional constraint:

$$\left\| \mathbf{x} \right\| = 1$$

- The problem becomes:

$$E(\mathbf{x}) = \left|\mathbf{Ux}\right|^2 = \mathbf{x}^T \mathbf{U}^T \mathbf{Ux}$$

$$\min(E(\mathbf{x})) \quad \text{s.t.} \left|\mathbf{x}\right| = 1$$

- U^TU is a symmetric positive semidefinite qxq matrix
- It can be diagonalized:

$$\mathbf{e}_i \quad i = 1,\ldots,q$$

$$0 \le \lambda_1 \le \ldots \le \lambda_q$$

$$\mathbf{x} = u_1\mathbf{e}_1 + \ldots + u_q\mathbf{e}_q$$

$$u_1{}^2 + u_2{}^2 + \ldots + u_q{}^2 = 1$$

$$E(\mathbf{x}) = \mathbf{x}^T U^T U \mathbf{x} = \lambda_1 u_1^{\ 2} + \ldots + \lambda_q u_q^{\ 2}$$

$$E(\mathbf{e}_1) = \mathbf{e}_1^{\ T} U^T U \mathbf{e}_1 = \lambda_1$$

$$E(\mathbf{x}) - E(\mathbf{e}_1) = \mathbf{x}^T U^T U \mathbf{x} - \mathbf{e}_1^{\ T} U^T U \mathbf{e}_1 =$$

$$= \lambda_1 u_1^{\ 2} + \ldots + \lambda_q u_q^{\ 2} - \lambda_1 \geq \lambda_1 (u_1^{\ 2} + \ldots + u_q^{\ 2} - 1) = 0$$

$$E(\mathbf{x}) \geq E(\mathbf{e}_1) = \lambda_1$$

The unit vector $\mathbf{x}$ minimizing $E(\mathbf{x})$ is the eigenvector $\mathbf{e_1}$ associated with the minimum eigenvalue of $\mathrm{U^T U}$.

The corresponding minimum value of E is λ_1

The problem can be solved using any technique for computing eigenvectors and eigenvalues. SVD in particular allows computing eigenvalues and eigenvector without constructing $U^T U$

Reconstruction of intrinsic and extrinsic parameters

Once the projection matrix *M* is estimated we can use its expression

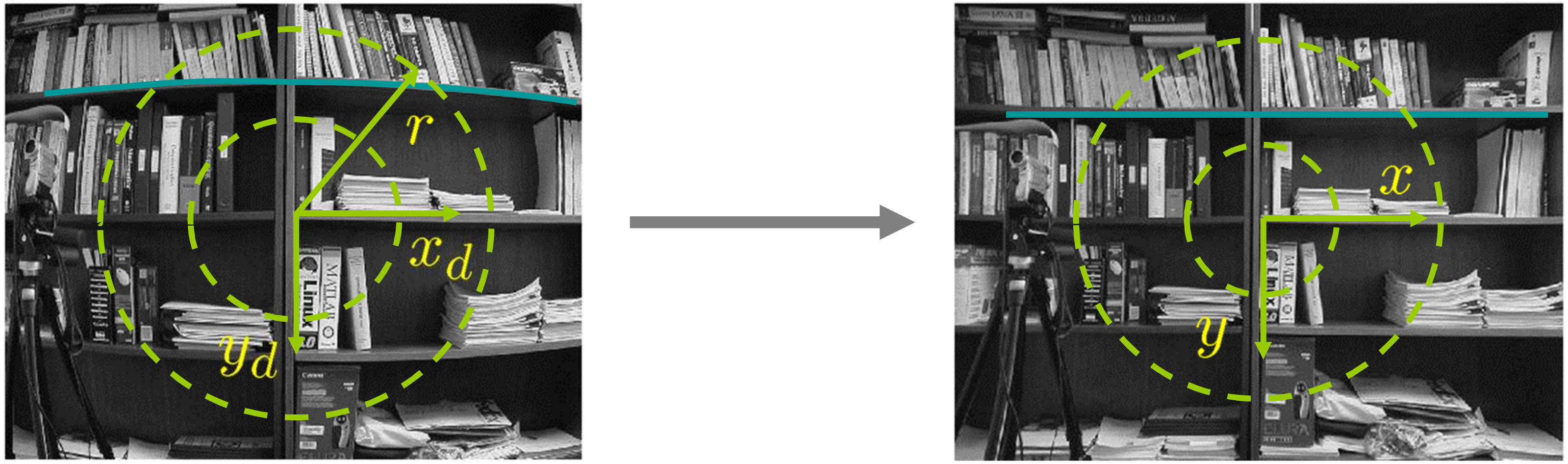
In a simple case in which theta=0, we get:

$$
\begin{aligned}
\mathbf{r}_3 &= m_{34}\mathbf{m}_3 \\
u_0 &= (\alpha\mathbf{r}_1^T + u_0\mathbf{r}_3^T)\mathbf{r}^3 = m_{34}^2\mathbf{m}_1^T\mathbf{m}_3 \\
v_0 &= (\beta\mathbf{r}_2^T + v_0\mathbf{r}_3^T)\mathbf{r}^3 = m_{34}^2\mathbf{m}_2^T\mathbf{m}_3 \\
\alpha &= m_{34}^2|\mathbf{m}_1 \times \mathbf{m}^3| \\
\beta &= m_{34}^2|\mathbf{m}_2 \times \mathbf{m}^3|
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{r}_1 &= \frac{m_{34}}{\alpha}(\mathbf{m}_1 - u_0\mathbf{m}_3) \\
\mathbf{r}_2 &= \frac{m_{34}}{\beta}(\mathbf{m}_2 - v_0\mathbf{m}_3) \\
t_z &= m_{34} \\
t_x &= \frac{m_{34}}{\alpha}(m_{14} - u_0) \\
t_y &= \frac{m_{34}}{\beta}(m_{24} - v_0)
\end{aligned}
$$

See Forsyth & Ponce for details and skew-angle case.

Radial distortion



$$x = x_d\left(1 + a_1 r^2 + a_2 r^4\right)$$

$$y = y_d\left(1 + a_1 r^2 + a_2 r^4\right)$$

Camera calibration becomes a non linear problem…